On the Value of Terrorist's Private Information in Government's Defensive Resource Allocation Problem

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Abstract

The ability to understand and predict the sequence of events leading to a terrorist attack is one of the main issues in developing pre-emptive defense strategies for homeland security. In this paper, we explore the value of terrorist's private information on government's defense allocation decision. In particular, we consider two settings with different informational structures. In the first setting, the government knows the terrorist's target preference but does not know whether the terrorist is fully rational in his target selection decision. In the second setting, the government knows the degree of rationality of the terrorist, but does not know the terrorist's target preference. We fully characterize the government's equilibrium budget allocation strategy for each setting and show that the government makes resource allocation decisions by comparing her valuation for each target with a set of thresholds. We then derive the value of information (VOI) from the perspective of the government for each setting. Our results show that VOI mainly depends on the government's budget and the degree of heterogeneity among the targets. In general, VOI goes to zero when government's budget is high enough. But, the impact of heterogeneity among the targets on VOI further depends on whether the terrorist's target preference matches with government's or not. Finally, we perform various extensions on the baseline model and show that the structural properties of budget allocation equilibrium still hold true.

Keywords: Homeland security; Value of information; Information asymmetry; Nonstrategic terrorist

1 Introduction

In defense budget allocation problem, the government (defender, from now on) often needs to prioritize and distribute her limited resources among valuable targets to defend them against an unpredictable terrorist (attacker, from now on) whose response strategy cannot be fully assessed a priori. Therefore, an important issue from the defender's point of view is the development of a defense strategy that allows her to incorporate this unpredictability into her budget allocation decisions. One of the main sources of unpredictability is related to the uncertainty in degree of rationality of the attacker in his target selection decisions. An extreme, though commonly assumed case, is one of a fully rational attacker who responds optimally to the defender's budget allocation decisions when he chooses which target to attack. However, this type of strategic behavior cannot fully explain some of the terrorist acts in the recent history. Indeed, the attack on the World Trade Center in New York and the four hijackings of September 11, 2001, suggest that the attackers may indeed be choosing their targets regardless of the observed defense levels. Note that, New York, where the attack happened, and Boston, where the hijackings happened, were both among top urban areas in terms of receiving US Homeland Security Grant Program between 2000 and 2008, based on the US Federal Funding Assistant (http://www.fedspending.org/). Furthermore, the target selection criteria of such nonstrategic attackers may be influenced by some other factors that are either not readily available to or difficult to observe for the defender. Therefore, the possibility that the attacker's degree of rationality and target selection criteria might come in many guises requires us to analyze various informational scenarios from the defender's point of view. In this paper, we aim to accomplish this goal in two stages. First, we develop an incomplete (asymmetric) information model to capture the different degrees of unpredictability in the attacker's response. Then, we explore this model to address the following research questions:

Research Question 1: How should a defender prioritize multiple targets and allocate limited budget among them when faced with two types of asymmetric information (degree of rationality and target preference) about the attacker?

Research Question 2: What is the impact of partial information on the defender's equilibrium budget allocation strategy?

Research Question 3: From the defender's perspective, what is the value of information regarding the attacker's degree of rationality and target preference? How does the value of information depend on problem parameters such as defender's budget, targets' valuations, and effectiveness of the defender's budget?

We develop a game-theoretic model considering the following sequence of decisions. First, the defender distributes her limited defense budget across given targets. The attacker responds to that by first choosing a target and then deciding on his effort level, which ultimately determines the degree of damage inflicted upon the selected target. Target selection by the attacker depends on whether he is strategic or not. A strategic attacker, who shares a common target valuation with the defender, chooses a target by taking into account both the valuation of that target and the defender's budget allocation decision, and responds optimally to defender's decision, whereas a nonstrategic attacker, whose target valuation is unknown to the defender, decides on the target by considering only his own valuation. In both strategic and nonstrategic cases, after the target is selected, the attacker then decides on his effort level. Finally, the damage on a target depends stochastically on the amount of budget allocated by the defender and the attacker's effort level. To address the above questions, we focus on two types of information asymmetry: (i) the attacker's degree of rationality and (ii) his target preference. Considering two scenarios (symmetric and asymmetric information) along each dimension, we build four models, as shown in Table 1.1.

Table 1.1: Models analyzed in this paper

		Rationality of attacker	
		Known	Unknown
Nonstrategic attacker's target preference	Known	Full symmetric information	Model A
	Unknown	Model B	Full asymmetric information

Addressing *first research question*, we characterize defender's equilibrium budget allocation under different informational scenarios depicted in Table 1.1. In particular, in each scenario above, we show that the equilibrium policy involves a set of thresholds for each target and that the defender makes her budget allocation decision by first ordering the targets with respect to a ranking rule, and comparing the valuations of targets with these thresholds. If the valuation of a target is sufficiently high, the defender invests non-zero budget to defend the target; otherwise, the target is left undefended.

To address our *second research question*, we examine the impact of asymmetric information on the ranking rule and thresholds. Our analysis shows that both the ranking rule and number and level of thresholds depend on the type of information available to the defender before she makes budget allocation decision. First, we show that in Model A (i.e., when the attacker's rationality is unknown to the defender), the defender employs valuation-based ranking rule and two thresholds for each target to decide whether the target should be defended or not, whereas in Model B (i.e., when the target preference is unknown to the defender) she uses belief-adjusted valuations to rank the targets and compare the valuation of each target with a single threshold to decide on her budget allocation. The reason behind this is that in Model

A, the defender must trade off the threat from a strategic attacker with that of a nonstrategic one. Since a strategic attacker differs from its non-strategic version in terms of its attack strategy, the defender has to simultaneously examine two thresholds (one for strategic and one for non-strategic) to hedge herself against risks from both of the types. However, in Model B, the defender has perfect information about the type of attacker. Consequently, she uses a single threshold to decide on her defense strategy.

Finally, to address our *third research question*, we characterize the value of information along each dimension by analytically comparing the defender's payoffs under Model A and B with those under symmetric information. We identify two phenomena that significantly influence the value of attacker's rationality information. First, the defensive strategies against a strategic and a nonstrategic attacker differ fundamentally. Against a strategic attacker, on one hand, the defender implements a comprehensive defense strategy in which she allocates her budget among all the targets in a way that will bring down the expected damage across all the targets to the same level. On the other hand, against a nonstrategic attacker, the defender ranks the targets from the highest to lowest value, adjusts the ranking by her a-priori information, and defends only a few of them with a more concentrated and focused defensive effort. Employing a comprehensive strategy against a nonstrategic attacker or a concentrated one against a strategic attacker leads to significant losses on the defender's side. The second factor is related to the difference between strategic and nonstrategic attackers' behaviors. Namely, a defender can influence a strategic attacker's behavior indirectly via her defensive budget allocations, whereas a similar strategy has limited influencing power on a nonstrategic attacker. Therefore, knowing the degree of rationality of the attacker gives significant prediction and control capability to the defender, which she can use to improve the security of the overall system. To summarize, these two effects suggest that additional value that can be obtained by knowing the degree of rationality of an attacker is more than that by knowing his target preference. In addition to the above three questions, we also explore how the rationality and target information behave with respect to system parameters, and show that it is sensitive to the total defensive budget, the effectiveness ratio of an attack, and the degree of heterogeneity among the targets.

2 Related Literature

Our paper is related to the growing body of literature that considers the role of incomplete information in defender-attacker games. This literature can be further divided into two streams, depending on whether it is the attacker or the defender who holds asymmetric information regarding his/her opponent. We refer the reader to Sandler and Siqueira [2009] for an extensive review of game-theoretical models in this literature.

The papers in the first stream consider the cases where the attacker is uncertain about the target properties and/or the defensive allocation. Since our focus is to analyze the impact of asymmetric information from the defender's perspective, we discuss only a few representative papers in this stream. Powell [2007a], Zhuang and Bier [2011], Zhuang et al. [2010] develop models in which the defender holds private information and explore whether the defender should hide this information from the attacker (secrecy policy) or convey a noisy signal (deception policy) to the attacker. There is also a body of literature exploring the impact of secrecy and deception in defense strategy. It assumes that the defender may be the first mover [Brown et al., 2005, Zhuang and Bier, 2011, Zhuang et al., 2010, Jenelius et al., 2010], the second mover [Overgaard, 1994, Brown et al., 2005, Arce and Sandler, 2007], or that the defender and attacker may play simultaneously [Hausken, 2010, Zhuang and Bier, 2007]. Jenelius et al. [2010] analyze the impact of the degree of an attacker information asymmetry on the likelihood of his attack and show that a less informed attacker may cause more damage to a defender. Kaplan et al. [2010] also consider the uninformed attacker setting and characterize a simple rule for the attacker's optimal strategy.

The second stream of research in this literature, which is more relevant to our model, studies the cases in which the defender has incomplete information regarding the attacker's attributes [Bier et al., 2007, Powell, 2007b, Bier et al., 2008, Rios and Insua, 2009, Wang and Bier, 2011, Rothschild et al., 2012, Nikoofal and Zhuang, 2012, Zhang and Ramirez-Marquez, 2012]. Specifically, Bier et al. [2007] study the defender's first-mover advantage when she publicly announces the defensive allocation rather than keeps it secret. They compare the equilibrium in sequential and simultaneous games when attacker's preferences are unknown by the defender. Their results show that, in equilibrium, the defender is always better off in the sequential game. Powell [2007b] explore the existence of Nash equilibrium for the defender's resource-allocation problem in different settings, including the case where the defender is unsure of the terrorists' preferred targets. Bier et al. [2008] analyze a sequential defender-attacker game, where the defender acts first and considers that the attacker's valuation for each target follows a two-parameter Rayleigh distribution with mean value equal to the defender's valuation. Nikoofal and Zhuang [2012] also study a defender-attacker game with incomplete information in which the attacker has private information about the valuation of targets. They use robust optimization techniques to model the defender's uncertainty about the attacker's attributes. Brown and Cox [2011] show that traditional probabilistic risk assessment can lead to poor defensive decisions when the attacker holds private information about his attack probabilities. Recently, Wang and Bier [2011] develop a two-period model for the defensive resource allocation problem, in which the attacker decides on the targets to attack based on a multiattribute utility function. In their model, they assume that the defender has prior beliefs on the attacker's attributes, which she updates upon observing the actions of the attacker. Finally, Hausken and Zhuang [2011] analyze government's decision to allocate its resources between attacking to downgrade a terrorist's resources and defending against a terrorist attack. In their model, the terrorist also allocates its resources between attacking a government's asset and defending its own resources. Our model differs from this stream in two ways: first, while papers in this stream commonly assumed that the attacker is fully rational and responds optimally to the defender's strategy, this paper models different degrees of rationality by considering two types of attacker behaviors: strategic and nonstrategic. Second, our paper's objective also differs from the above papers'. Most of the previous literature focuses only on one-dimensional information asymmetry (mostly target preference), and analyze its impact on the defender's budget allocation decision. In contrast, our work considers two types of information asymmetry (both target preference and degree of rationality of the attacker).

Our paper is also related to a series of papers [Zhuang and Bier, 2007, Powell, 2007b, Levitin and Hausken, 2009, Golany et al., 2009, Hao et al., 2009, Hausken et al., 2009, Shan and Zhuang, 2013a] that consider both terrorist and non-terrorist attacks (natural disasters, for example). Do note however that our definition of a nonstrategic attacker is different from the definition of a non-terrorist attacker used in the above models. Indeed, a nonstrategic attacker can still adjust his effort endogenously in response to the defender's budget allocation decision, whereas a non-terrorist attacker is modeled in the above papers as a passive actor whose level of effort entirely depends on exogenous factors. The work that comes closest to ours in this context is Shan and Zhuang [2013a], in which, similar to our study, the terrorist might be strategic or non-strategic. To model the non-strategic attacker's behavior, Shan and Zhuang [2013a] assume that such an attacker attacks with an exogenously-determined probability. Our paper differs from Shan and Zhuang [2013a] in different ways. First, we assume that the non-strategic attacker keeps different valuation of the targets than the government. We operationalize the valuation difference between non-strategic attacker and the government by creating different types for the non-strategic terrorist. Second, our definition of non strategic attacker differs from that in Shan and Zhuang [2013a], in the sense that the non-strategic attacker chooses its target irrespective of defensive budget allocation, but he can update his attack level based on the defense level observed on his preferred target, hence we develop a game-theoretical model to capture non-strategic attacker decision. Finally, our modeling approach for the non-strategic attacker helps us to answer one of the main research questions of this paper, which is to explore the value of terrorist's private information in government defensive resource allocation.

3 Model Framework

To address the research questions raised in §1, we develop a two-stage non-zero-sum game between defender and attacker. In the first stage, the defender distributes her limited budget D across N different targets. Let v_i and D_i be the defender's valuation and budget allocation for target i, respectively. In the second stage, the attacker selects a target to attack and the level of effort to exert. Let A_i be the effort exerted by the attacker for target i. Once decisions over D_i and A_i are made, target i's damage is realized.

We use a likelihood function $P(D_i,\, A_i)$ to capture the expected damage for target i as a joint function of defender's budget allocation decision D_i and attacker's effort A_i. We assume that $P(D_i,\, A_i)$ increases in A_i, and decreases in D_i. Also, it has the following regularity properties: (i) $P(D_i,\, A_i)$ is twice differentiable with respect to A_i and D_i and (ii) $\lim_{A_i \to 0} P(D_i,\, A_i) = 0$, and $\lim_{D_i \to \infty} P(D_i,\, A_i) = 0$. An appropriate candidate for the damage probability function that parsimoniously satisfies the above properties is the cumulative exponential function [Bier et al., 2008, Gerchak and Safayeni, 1996, Golalikhani and Zhuang, 2011, Shan and Zhuang, 2013b].

$$P(D_i,\, A_i) = 1 - \exp\left(\frac{-\lambda A_i}{D_i}\right) \tag{3.1}$$

where λ is effectiveness ratio of an attack and measures the sensitivity of the damage function with respect to attacker's effort level per unit of budget investment [Bier et al., 2008, Wang and Bier, 2011]. To avoid additional complexity, without loss of generality, we assume $\lambda \in [0,\, 1]$.

In order to model the defender's incomplete information on the attacker's degree of rationality, we use Harsanyi's transformation to create two types for the attacker: strategic (denoted by s-type) and nonstrategic (denoted by n-type). Let p_s and p_n be the defender's a-priori beliefs about the true type of attacker, where $p_s + p_n = 1$. The s-type (strategic) attacker has the same target valuations as the defender and responds to the defender's budget allocation decision optimally. Specifically, if the s-type attacker wants to make an attack on target i, he chooses attack effort A_i^s that maximizes his expected payoff:

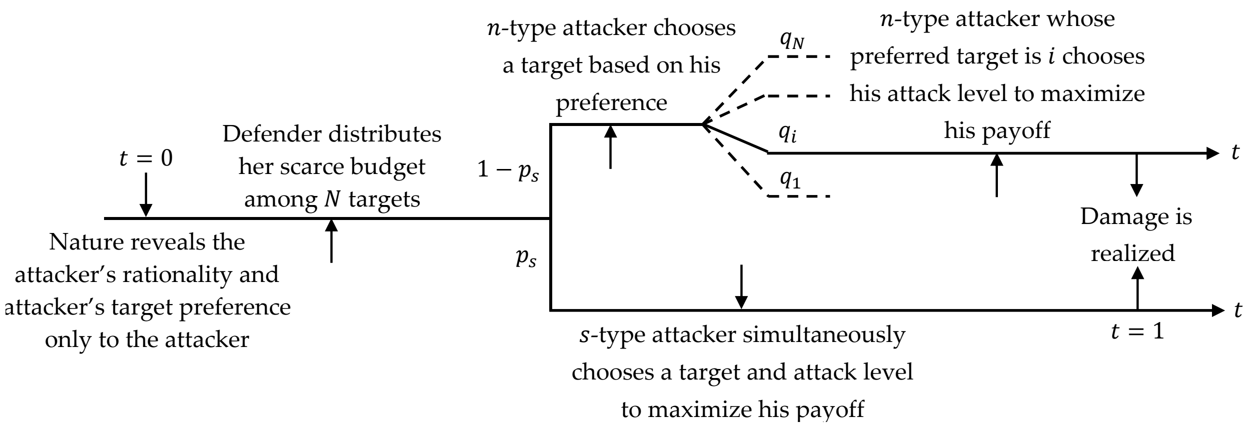
$$\pi_i^s\,(A_i^s) = \max_{A_i^s \geq 0} v_i P(D_i,\, A_i^s) - A_i^s \tag{3.2}$$

The s-type attacker then chooses target i^* that provides him with the maximum payoff among the N targets,

$$i^* = \arg\max_{i=1,\ldots,N} \pi_i^s\,(A_i^s) \tag{3.3}$$

The n-type (nonstrategic) attacker differs from the s-type attacker in two ways. First, he chooses his target without considering the defender's budget allocation decision. Second, his target preference is unknown to the defender. To model these two features in a tractable way, we use again Harsanyi's transformation to create N different n-type attacker for each target $i = 1, \ldots, N$ and denote the type that attacks target i irrespective of defender's budget allocation decision by n_i-type. If the attack is successful, the attacker obtains a fixed value of w, where w is a positive number large enough so that n_i-type can never be deterred from attacking his preferred target i. Let q_i, $i = 1, \ldots, N$ denote the defender's a-priori belief for the n_i-type attacker, where q_i is common knowledge and $\sum_{i=1}^{N} q_i = 1$. Even though n_i-type attacker's target selection decision does not depend on the defender's budget allocation decision, his effort level decision (denoted

Figure 3.1: Timing of Events



by A_{n_i}) does indirectly depend on it in the sense that his expected payoff is as follows,

$$\pi_{n_i} = \max_{A_{n_i} \geq 0} wP(D_i, A_{n_i}) - A_{n_i} \tag{3.4}$$

The defender makes her budget allocation decision to minimize the total expected damage subject to budget and non-negativity constraints:

$$\text{(Model I)} \quad \min \quad p_s \sum_{i=1}^{N} v_i P(D_i, A_i^s) + (1 - p_s) \sum_{i=1}^{N} q_i v_i P(D_i, A_{n_i}) \tag{3.5}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} D_i \leq D \tag{3.6}$$

$$D_i \leq v_i, \; i = 1, ..., N \tag{3.7}$$

$$D_i \geq 0, \; i = 1, ..., N \tag{3.8}$$

where constraints (3.6) and (3.7) ensure that the total budget allocation to all targets is less than the defender's maximum budget, and that the defender never invests on a target more than its valuation. The timing of events is shown in Figure 3.1.

We use the *Bayesian Nash Equilibrium* (BNE) solution concept to analyze the above game. In our modeling framework, BNE consists of a collection of budget allocations for the defender and type-contingent efforts for the attacker that are required to yield a perfect Nash equilibrium for the two-stage game between the defender (leader) and s- and n_i-type attackers (followers). For a detailed definition of BNE, please refer to Fudenberg and Tirole [1991].

We begin our analysis of Model I by characterizing the attacker's type-contingent best-response function. The best-response function for s-type attacker, denoted by $R^s(D_{i^*})$, can be obtained by solving the first order condition for his payoff function (3.2) with respect to his effort as follows:

$$R^s(D_{i^*}) = \begin{cases} 0 & \text{if } D_{i^*} \geq \overline{D}_{i^*} \\ \frac{D_{i^*}}{\lambda} \ln\left(\frac{\lambda v_{i^*}}{D_{i^*}}\right) & \text{otherwise} \end{cases} \tag{3.9}$$

where i^* is the target that provides the s-type attacker with the maximum payoff among the N targets (Eq. 3.3). Note that $R^s(D_{i^*})$ initially increases with D_{i^*}, when $0 < D_{i^*} \leq \frac{\lambda v_{i^*}}{e}$ then decreases with D_{i^*} when $\frac{\lambda v_{i^*}}{e} \leq D_{i^*} < \lambda v_{i^*}$, and finally is equal to zero for $D_{i^*} \geq \lambda v_{i^*}$. Hence, $\overline{D}_{i^*} = \lambda v_{i^*}$ is the lowest possible level of defense required to deter s-type attacker from attacking target i. Next, we consider the best-response function of the n_i-type attacker. Similarly, obtaining the first order condition from Equation (3.4) and solving it for A_{n_i} provide us with the n_i-type attacker's best-response function as follows:

$$R^{n_i}(D_i) = \frac{D_i}{\lambda} \ln\left(\frac{\lambda w}{D_i}\right) \tag{3.10}$$

Recall that we assume that n_i-type attacker's target selection decision cannot be deterred by the defender's budget allocation decision. This implies that $R^{n_i}(D_i)$ should be positive for all feasible values of budget allocation, which imposes the following condition on target valuations and λ:

Assumption 1. $w > \max\limits_{i=1,\dots,N} \{v_i\} / \lambda$.

Finally, substituting both n_i- and s-type attackers' best-response functions into the defender's optimization problem gives us the following optimization problem:

$$\min \quad p_s \sum_{i=1}^{N} v_i \left(1 - \frac{D_i}{\lambda v_i}\right) I_{i=i^*} + (1-p_s) \sum_{i=1}^{N} q_i v_i \left(1 - \frac{D_i}{\lambda w}\right) \tag{3.11}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} D_i \leq D \tag{3.12}$$

$$D_i \leq v_i,\ i = 1, \dots, N \tag{3.13}$$

$$D_i \geq 0,\ i = 1, \dots, N \tag{3.14}$$

where $I_{i=i^*}$ is the binary indicator variable that takes 1 if $i^* = \arg\max\limits_{i=1,\dots,N}\{v_i P(D_i, A_i^s) - A_i^s\}$ and 0 otherwise. Note that in our game theoretical model, s- and n_i-type attackers maximize the expected damage on their selected targets (target i^* for s-type and target i for n_i-type), while the defender minimizes it. Hence, this leads to an equilibrium budget allocation for the defender that minimizes the attacker's maximum payoff. Using this observation, we can get rid of the binary indicator variables $I_{i=i^*}$ by introducing a variable z that bounds the maximum payoff for the s-type attacker:

$$\text{(Model II)} \quad \min \quad p_s z + (1-p_s)\sum_{i=1}^{N} q_i v_i \left(1 - \frac{D_i}{\lambda w}\right) \tag{3.15}$$

$$\text{s.t.} \quad v_i \left(1 - \frac{D_i}{\lambda v_i}\right) \leq z,\ i = 1, ..., N \tag{3.16}$$

$$\sum_{i=1}^{N} D_i \leq D \tag{3.17}$$

$$D_i \leq v_i,\ i = 1, ..., N \tag{3.18}$$

$$D_i \geq 0,\ i = 1, ..., N \tag{3.19}$$

Note that Model II is a linear programming problem with respect to D_i. Therefore, to characterize the defender equilibrium strategy, we can apply the optimality principle of linear programming, which effectively states that when the feasible set is nonempty and bounded, then at least one optimal solution is located at an extreme point [Dantzig, 1951]. Some of the constraints in Model II are therefore binding at the optimal extreme point, which enable us to determine the basic variables. Specifically, assuming that constraint (3.16) is binding for some of the targets, namely, $i \in I_D$, we have $D_i = \lambda(v_i - z)$, $i \in I_D$, and $D_i = 0$, $i \notin I_D$. Replacing these values in Model II gives a tractable model to characterize the defender's equilibrium strategy. Before analyzing the equilibrium strategy using this methodology, we summarize the list of notation used for the problem parameters and decision variables in Table 3.1.

Table 3.1: Notations and decision variables

Notations	
N	Number of targets
p_s	The proportion of s-type attackers
q_i	The proportion of n-type attackers whose preferred target is i; $\sum_{i=1}^{N} q_i = 1$
D	Total budget of the defender
v_i	Defender's and s-type attacker's valuation of target i
w	The valuation of n-type attacker's preferred target
λ	Effectiveness ratio of an attack
$P(D_i, A_i)$	Likelihood of damage function given budget allocation D_i and level of attacker's effort A_i
Decision variables	
D_i, A_i^s, A_i^n	Defender's budget allocation, s- and n-type attackers' efforts on target i

4 Allocation Equilibrium under Symmetric Information

In this section, we characterize the defender's equilibrium strategy under symmetric information scenario, where the defender knows both the attacker's degree of rationality and target preference. The following proposition provides the symmetric information equilibrium:

Proposition 1. *In equilibrium,*

1. *[Prioritization of the targets] The defender prioritizes all the targets in decreasing order with respect to their valuations v_i.*

2. *[Distribution of budget] The defender distributes budget to the i^{th} most valuable target if and only if its valuation v_i exceeds a threshold t_i where $t_i = \frac{\lambda \sum_{j=1}^{i} v_j - D}{i\lambda}$.*

3. *[Budget allocation] The defender's optimal budget allocation decision is fully characterized by Algorithm 1 provided in the Proof.*

Proof. First, assume that the attacker is of s-type. So, let $p_s = 1$ in Model II. Clearly, Model II is a linear optimization problem in terms of D_i and z. Therefore, the optimal solution is an extreme point of feasible region, at which constraint 3.16 is binding for some of the targets. Let I_D and I_{ND} indicate the set of targets for which constraint 3.16 is binding and non-binding, respectively. This implies that $D_i = \lambda(v_i - z)$, $\forall i \in I_D$, and $D_i = 0$, $\forall i \in I_{ND}$. By plugging these values into the objective function, the defender's optimization problem can be rewritten as follows:

$$\min \quad z \tag{4.1}$$

$$\text{s.t} \quad v_i \leq z,\ i \in I_{ND} \tag{4.2}$$

$$\sum_{i \in I_D} \lambda(v_i - z) \leq D \tag{4.3}$$

Note that the above model is a linear optimization model with respect to z. Since the s-type attacker's valuation of targets is the same as defender's, the defender should first protect against the most valuable target. The more the defender spends on the most valuable target, the smaller the attacker's payoff function will be until the strategic attacker maximizes his payoff by attacking to the second most valuable target. The defender should now protect both targets to make the success probability of the attack for both targets as small as possible such that the strategic attacker maximizes his payoff by hitting the third most valuable target. The defender keeps distributing the budget in this way to make a set of most valuable targets less and less damageable until she fully allocates her limited budget, or all targets being fully defended. The above discussion yields the following solution approach: Let us first order the targets with respect to their valuations such that $i = 1$ indicates the most valuable target and $i = N$ shows the least one. We can then reduce constraint set 4.2 to only one constraint; $\max_{i \in I_{ND}} \{v_i\} \leq z$. That means, given a partition of target into I_D and I_{ND}, the objective function will be limited by the maximum valuation in undefended subset of the targets, i.e., $\max_{i \in I_{ND}} \{v_i\}$. In order to reduce the objective function further, the defender has to pick the most valuable target from I_{ND} and allocate optimal budget to defend it as long as her total budget constraint 4.3

is satisfied. The optimal solution therefore is of a threshold-type policy, where targets are added to set I_D in order of their values until either the constraint 4.3 becomes binding, or all targets are being defended. We summarize our proposed approach in the following Algorithm 1 in order to characterize the defender's equilibrium.

Algorithm 1 (Defense equilibrium under symmetric information):

1. Order the targets in decreasing order of their values such that $i = 1$ indicates the most valuable target and $i = N$ shows the least valuable target. Initialize $k = 1$.
2. Calculate $b_k = \frac{\lambda \sum_{i=1}^{k} v_i - D}{\lambda k}$ for each $k \in \{1, \ldots, N-1\}$, and stop first time when $b_k \geq v_{k+1}$. Then, it is not optimal to defend targets whose valuations are less than or equal to v_{k+1}. The optimal defense allocation is $D_i = \lambda(v_i - b_k)$, $i \leq k$, and $D_i = 0$, $i \geq k+1$.
3. If $b_k < v_{k+1}$ for all $k \in \{1, \ldots, N-1\}$, then it is optimal to defend all the targets. The optimal defense allocation is then $D_i = \lambda v_i$, $\forall i$.

Note that by using the above Algorithm, we can define a threshold $t_i = \frac{\lambda \sum_{j=1}^{i} v_j - D}{i\lambda}$ for each target i such that target i would be defended in equilibrium as long as $v_i \geq t_i$. Now, assume that the attacker is of n-type, i.e., $p_s = 0$ in Model II. The problem then boils down to a simple knapsack problem. Since the defender knows the n-type attacker's preferred target, namely target i, then the defender would minimize her objective function by allocating the maximum budget to target i subject to feasibility constraints 3.17, 3.18 and 3.19, i.e., $D_i = \min\{v_i, D\}$. □

Note that from the threshold expression in Proposition 1, when the defender faces with an s-type attacker, the likelihood that she defends target i increases with both target value, v_i, and defender's budget, D, but decreases with λ. Let us describe the underlying rationale behind Proposition 1. First of all, it is obvious that the most valuable target is also the target that will be protected first by the defender. However, the more the defender allocates budget to protect this target, the less the attacker's payoff will be for it, until the attacker becomes better off by switching to the second most valuable target. The defender should now protect both targets, and allocate budget to them so that the attacker's payoff is the same across both targets. The defender continues in this way to make a subset of the most valuable targets less and less exposed while ensuring that no target suffers more damage than any other one. Therefore, the equilibrium thresholds characterized in Proposition 1 are determined just to equalize the expected damage across all the defended targets. In the next section, we use the same approach to characterize the defense equilibrium when the defender has only partial information about the attacker's attributes.

5 Allocation Equilibrium under Asymmetric Information

In the previous section, we showed how the defender optimally distributes her scarce budget among the targets when exact information about the attacker's attributes is available to her. In this section, we consider the cases where the defender is uncertain about one of these attributes (degree of rationality in §5.1 and target preference in §5.2) of the attacker. Specifically, we answer the following two questions: (i) How should defender prioritize targets for budget allocation under asymmetric information? (ii) How should she distribute budget among those targets that are to be defended?

5.1 Information Asymmetry about an Attacker's Degree of Rationality (Model A)

The aim of this section is to investigate the situation where the defender knows precisely the target preference of the non-strategic attacker, but has partial information about his true degree of rationality. More specifically, she has a-priori beliefs that p_s proportion of the attackers are s-type and $p_n = 1 - p_s$ of them are n_k-type, where k is the preferred target of non-strategic attackers that is known to the defender. Proposition 2 summarizes the defender's equilibrium in this case (note that, from now on, all proofs are provided in the Appendix):

Proposition 2. *There exists a BNE in which*

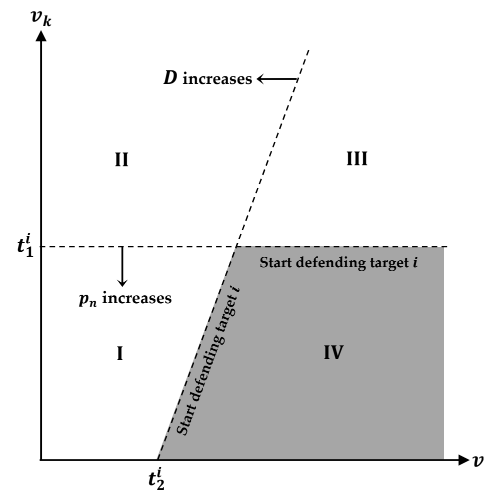
1. *[Prioritization of the targets] The defender prioritizes all the targets (except the target preferred by n_k-type attacker) in decreasing order with respect to their valuations v_i.*

2. *[Distribution of budget] The defender distributes budget to the i^{th} most valuable target if and only if the following two conditions are satisfied: $t_1^i \geq v_k$ and $v_i \geq t_2^i$, where $t_1^i = \frac{p_s w}{i(1-p_s)}$, and $t_2^i = \frac{\lambda \sum_{j=1}^{i} v_j + \lambda v_k - D}{\lambda(i+1)}$. Furthermore, let S be the subset of all defended targets except target k, i.e., $S = \{i \mid t_1^i \geq v_k,\ v_i \geq t_2^i\}$. Then, target k is defended if and only if $v_k > \frac{\lambda \sum_{i \in S} v_i - D}{\lambda |S|}$.*

3. *Finally, the optimal defense allocation is $D_i^* = \lambda(v_i - \mathscr{B})$, $i \in S$, $D_i^* = 0$, $i \notin S$, and $D_k^* = \min\{[D - \sum_{i \in S} [\lambda(v_i - \mathscr{B})]^+]^+, v_k\}$, where $\mathscr{B}$ is characterized in Appendix (see Algorithm 2).*

Note that even though prioritization scheme utilized in equilibrium under asymmetric information is same as that under symmetric one, the budget distribution policy is different. Specifically, there are two thresholds in asymmetric setting (as opposed to one threshold in symmetric information setting) that need to be checked by the defender in order to distribute non-zero budget to the i^{th} most valuable target. This implies that the conditions under which a target is defended become more restrictive under asymmetric information (as characterized in proposition 2) than under symmetric information (as characterized in Proposition 1). The rationale behind this difference relies on the fact that under asymmetric information,

Figure 5.1: Defender's optimal decision under uncertainty about attacker's rationality information

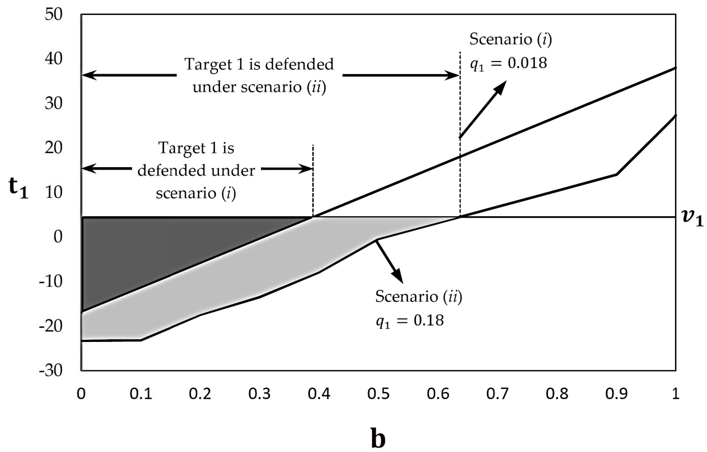


the defender needs to take into account the threats from not only an s-type attacker, (by comparing v_i's with t_2^i), but also an n_k-type attacker (by comparing v_k and t_1^i). In other words, the budget distribution for the i^{th} most valuable target depends on not only the characteristics of its own and targets of higher valuation (via t_2^i) but also that of a special target k (via t_1^i). Note also that the threshold t_1^i is the highest for the most valuable target and decreases as the target becomes less valuable. The joint effects of two thresholds on the budget allocation decision for the i^{th} most valuable target are shown in Figure 5.1. As highlighted by region IV, target i is defended only when the expected threat from n_k-type attacker is relatively low and its valuation is sufficiently high. Also note that the likelihood of target i being defended decreases as $p_n = (1 - p_s)$ increases. This is because when the threat from an n_k-type attacker increases, the defender shifts resources away from the targets that can potentially be attacked by an s-type to target k (i.e., n_k-type attacker's preferred target).

5.2 Information Asymmetry about Nonstrategic Attacker's Target Preference (Model B)

Now in this section, we assume that attacker's type is known by the defender. If true type is s-type, the budget allocation problem facing the defender becomes similar to the symmetric information game considered in §4. Note that, in §7.1, we extend the analysis when defender and s-type attacker share different target valuations. Hence, in this section, we only consider the case where the true type of attacker is n-type and the only information asymmetry is on the attacker's preferred target. To model this type of information asymmetry, we assume that the defender has a-priori beliefs that the q_i proportion of (non-strategic) attackers attacks target i. Next proposition characterizes the defender's equilibrium budget allocation under this

Figure 5.2: Optimal threshold to defend the target with the smallest valuation when $N = 10$, $a = 1$, and $0 \leq \mathtt{b} \leq 1$



scenario:

Proposition 3. *Assume that the attacker is n_i-type and his preferred target is unknown to the defender. Then, there exists a BNE in which*

1. *[Prioritization of the targets] The defender prioritizes targets in decreasing order with respect to $q_i v_i$, and*
2. *[Distribution of budget] The defender distributes budget to the i^{th} most valuable target (in the sense of $q_i v_i$) if and only if $v_i \geq t_i$, where $t_i = \sum_{j=0}^{i} v_j - D$, and $v_0 = 0$.*

Note that in Proposition 3, the defense strategy also consists of a single threshold (as in symmetric information case). However, the defender changes her prioritization scheme by ordering the targets based on not only their valuations but also her a-priori information. To illustrate how this new prioritization scheme affects the budget allocation decision, we consider two different scenarios depending on whether a-priori beliefs are correlated with target valuations or not. Specifically, in the first scenario (scenario (i)), we consider $v_i = 1 + \mathtt{b}i$, and $q_i = \frac{i}{\sum_{j=1}^{N} j}$, while in the second one (scenario (ii)), we assume that $v_i = 1 + \mathtt{b}i$, and $q_i = \frac{N-i+1}{\sum_{j=1}^{N} j}$. Note that in scenario (i) (resp., scenario (ii)), the valuation of target i is positively (resp., negatively) correlated with a-priori beliefs. Also, in both scenarios, $\mathtt{b}$ measures the degree of heterogeneity among target valuations. Arranging the targets in descending order with respect to $q_i v_i$ and comparing v_i with t_i as characterized in Proposition 3, we can determine whether target i is defended or not. For illustrative purposes, we consider only the optimal threshold for Target 1 (i.e., the target with smallest valuation) and numerically analyze how it behaves under each scenario with respect to $\mathtt{b}$ in Figure 5.2.

We can make two observations from Figure 5.2. First, under both scenarios, t_1 increases in $\mathtt{b}$, i.e., the likelihood of Target 1 being defended decreases as $\mathtt{b}$ increases. This is due to the fact that as the degree of target heterogeneity increases, the gap between the targets with smallest and highest valuations widens, which enforces the defender to shift more resources toward higher valuation targets. Second, the negative correlation between a-priori beliefs and target valuations makes the targets equally valuable from the defender's perspective according to $q_i v_i$-ranking hence it smoothens budget allocation among the targets. This is why the likelihood of Target 1 being defended under scenario (ii) is more than that under scenario (i).

6 Value of Attacker's Information

In §5, we characterized the defender's equilibrium budget allocation when she makes her decision without knowing either (i) true degree of rationality (Model A), or (ii) target preference (Model B) of an attacker. In this section, *addressing research question 3*, we explore how much the defender can potentially gain if she could make her decisions after she observes either one of these attacker's characteristics, and how this additional gain due to information depends on problem data such as targets valuations, v_i, effectiveness ratio of attack, λ, and maximum budget, D. In general, when the attackers and defenders interact repeatedly over time then the defender has opportunity to update her beliefs about the attacker's attributes. In reality, the attacker may arrange numerous trials to evaluate the defense level before making the final attack. For example, the attacker may make successive attacks (as the series of attacks by Al-Qaeda in the United States to World Trade Center in February 26, 1993 and September 11, 2001) or engage in successive attempts to probe a system before a successful attack (as in the case of computer security [Zhuang et al., 2010]). Under such scenarios, the defender can observe attacker's choice of target and level of effort, and thus update her beliefs about attacker's attributes for possible attack in the future. To define the value of information, we compare the defender's payoffs under symmetric and asymmetric settings. For analytical simplicity, we assume that there are only two targets k and k' where target k is more valuable than target k', i.e., $0 < v_{k'} \leq v_k \leq 1$. Recall that the maximum payoff the defender can generate is always under symmetric information setting, where the defender knows both characteristics of the attacker before allocating her budgets. On the other hand, under asymmetric information, the defender makes her budget allocation decision without knowing one of these characteristics. Therefore, the value of information in a defender-attacker game can be defined as the difference between the defender's expected payoffs under symmetric and asymmetric information scenarios. In our setting, it translates into the difference between the defender's payoffs under the symmetric (characterized in Proposition 1) and asymmetric (characterized in Proposition 2 for rationality and Proposition 3 for target preference) scenarios. Let $\mathbb{V}_R^i$ and $\mathbb{V}_T^i$ be the value of rationality and target preference information, respectively, when the attacker is i-type, where $i = s, n_k, n_{k'}$. Let us first establish the value of rationality information.

6.1 Value of Attacker's Rationality Information

Recall that the characterization of the value of rationality information (VOR) depends on the true type of the attacker (s- or n-type) as well as on the preferred target of the n-type attacker (target k or k'). We can therefore obtain four different cases, for which we characterize VOR as follows:

Proposition 4. *The value of rationality information (VOR) is fully characterized in Table 6.1 (when the true type is n) and Table 6.2 (when the true type is s).*

Note that in each case VOR depends on two main parameters: the defender's a-priori beliefs about the attacker's type and the defender's budget. Before discussing the impact of these and other parameters on VOR, we note that VOR is equal to zero if either the true type is s and $p_s > t_s$ in Table 6.2, or the true type is n and $p_s < t_n$ in Table 6.1. In other words, obtaining rationality information has no value to the defender if her prior beliefs correctly match with the true type of the attacker. Therefore, in what follows, we focus only on the cases where these two do not match. Those are the cases where the degree of information asymmetry between the defender and attacker is sufficiently high, i.e., $p_s \geq t_n$ in Table 6.1 and $p_s < t_s$ in Table 6.2.

Table 6.1: Value of rationality information when the attacker is n-type

	Condition on defender's beliefs	Condition on defender's budget	VOR
Non-strategic attacker is n_k-type	$p_s < \bar{t}_n$	Any value of D	0
	$p_s \geq \bar{t}_n$	$v_k + \lambda v_{k'} < D$	0
		$v_k + \lambda v_{k'} \geq D$, $v_k < D$,$\lambda(v_k + v_{k'}) < D$	$v_k - D + \lambda v_{k'}$
		$v_k < D$,$\lambda(v_k + v_{k'}) \geq D$	$v_k - \frac{D+\lambda\Delta v}{2}$
		$v_k \geq D$,$\lambda(v_k + v_{k'}) < D$	$\lambda v_{k'}$
		$v_k \geq D$,$\lambda(v_k + v_{k'}) \geq D$, $\lambda\Delta v < D$	$\frac{D-\lambda\Delta v}{2}$
		$v_k \geq D$,$\lambda(v_k + v_{k'}) \geq D$, $\lambda\Delta v \geq D$	0
Non-strategic attacker is $n_{k'}$-type	$p_s < \underline{t}_n$	Any value of D	0
	$p_s \geq \underline{t}_n$	$v_{k'} + \lambda v_k < D$	0
		$v_{k'} + \lambda v_k \geq D$, $v_{k'} < D$,$\lambda(v_k + v_{k'}) < D$	$\kappa_v(v_{k'} - D + \lambda v_k)$
		$v_{k'} \geq D$,$\lambda(v_k + v_{k'}) < D$	$\lambda v_{k'}$
		$v_{k'} < D$,$\lambda(v_k + v_{k'}) \geq D$, $\lambda\Delta v < D$	$\kappa_v(v_{k'} - \frac{D}{2} + \lambda\Delta v)$
		$v_{k'} \geq D$,$\lambda(v_k + v_{k'}) \geq D$, $\lambda\Delta v < D$	$\kappa_v \frac{D+\lambda\Delta v}{2}$
		$v_{k'} \geq D$,$\lambda(v_k + v_{k'}) \geq D$, $\lambda\Delta v \geq D$	$\kappa_v D$
		$v_{k'} < D$,$\lambda(v_k + v_{k'}) \geq D$, $\lambda\Delta v \geq D$	$\kappa_v v_{k'}$

Notes. $\bar{t}_n = \frac{\lambda}{1+\lambda}$; $\underline{t}_n = \frac{\lambda v_{k'}}{v_{k'}+\lambda v_{k'}}$; $\Delta v = v_k - v_{k'}$; $\kappa_v = \frac{v_{k'}}{v_k}$

- [The impact of defender's budget] In general, VOR initially increases then decreases in the defender's

Table 6.2: Value of the attacker's rationality information when the attacker is s-type

	Condition on defender's beliefs	Condition on defender's budget	VOR
Nonstrategic attacker is n_k-type	$p_s < \bar{t}_s$	$\lambda(v_k + v_{k'}) < D$	$v_{k'}$
		$\lambda(v_k + v_{k'}) \geq D > \lambda\Delta v$	$\frac{D-\lambda\Delta v}{2\lambda}$
		$D \leq \lambda\Delta v$	0
	$p_s \geq \bar{t}_s$	Any value of D	0
Nonstrategic attacker is $n_{k'}$-type	$p_s < \underline{t}_s$	$\lambda(v_k + v_{k'}) < D$	v_k
		$\lambda(v_k + v_{k'}) \geq D$	$\frac{D+\lambda\Delta v}{2\lambda}$
	$p_s \geq \underline{t}_s$	Any value of D	0

Notes. $\bar{t}_s = \frac{\lambda}{1+\lambda}$; $\underline{t}_s = \frac{\lambda v_{k'}}{v_{k'}+\lambda v_{k'}}$; $\Delta v = v_k - v_{k'}$

budget D. Recall from Proposition 1 that the defender's budget allocation decision under symmetric information depends on the true type of attacker. That is, if the true type is of n, she only defends the preferred target of the n-type attacker, whereas if the true type is of s, she distributes her budget among the most valuable targets proportionally so that these targets are protected equally well. However, the analysis of Model A shows that the defender uses both strategies in order to hedge herself against both s- and n-type attackers. As a result, due to the lack of rationality information, the defender ends up wasting money on the targets that are less likely to be attacked when her budget is limited. This implies that the VOR initially increases in D for small values of D. However, as D keeps increasing (see column "condition on defender's budget" in Table 6.1 from bottom to top), even though the defender wastes some money on the less vulnerable targets, she still has sufficient funds to defend all the targets, which makes rationality information less of an issue for the defender.

- [The impact of λ] The impact of λ on VOR depends on whether defender's and attacker's target preferences match. In general, VOR decreases in λ if they match; otherwise, it decreases. The rationale behind the above observation is that λ impacts the defender's budget differently depending on whether the true type is n or s. As shown in Proposition 1, under symmetric information, the defender adjusts her budget allocation using λ if the attacker is s-type, whereas her budget allocation does not depend on λ if the attacker is n-type. This means, ceteris paribus, that if the true type is s, VOR decreases with λ because under both symmetric and asymmetric information, the defender spends money only on the most valuable targets as λ increases. On the other hand, the impact of λ on VOR when the true type is n is further complicated by whether the n-type attacker's target preference matches with the defender's or not. If they match, then VOR decreases with λ, otherwise it increases with λ.

- Finally, we discuss the sensitivity of VOR with respect to the degree of heterogeneity in valuation between targets k' and k, i.e., $\Delta v = v_k - v_{k'}$. The result in this case also depends on the true type and the degree of mismatch between target preferences of the defender and attacker. In general, VOR increases with Δv when target preferences match otherwise it decreases in Δv.

6.2 Value of Target Information

We now consider the value of the target information (VOT) given that the defender knows the true type of the attacker. In contrast to the value of rationality information (VOR), VOT has only two sub-cases because VOT is zero when the type of the attacker is s. Therefore, in the following proposition, we characterize VOT only when the attacker is of n-type.

Proposition 5. *The value of the nonstrategic attacker's target information is fully characterized in Table 6.3.*

Similar to our discussion about VOR information, VOT is also non-zero only when the degree of information asymmetry between the defender and the n-type attacker is sufficiently high, i.e., when either $q_k \geq Q$ and the true target is k', or $q_k \leq Q$ and the true preferred target is k. Below, we briefly discuss the impact of D, λ, and Δv on VOT:

Table 6.3: Value of nonstrategic attacker's target information

<table>
<tr><th></th><th>Condition on defender's beliefs</th><th>Condition on defender's budget</th><th>VOT</th></tr>
<tr><td rowspan="5">Nonstrategic attacker is n_k-type</td><td rowspan="4">$q_k < Q$</td><td>$v_k + v_{k'} < D$</td><td>0</td></tr>
<tr><td>$v_{k'} < D; v_k < D; v_k + v_{k'} \geq D$</td><td>$v_k + v_{k'} - D$</td></tr>
<tr><td>$v_{k'} < D; v_k \geq D$</td><td>$v_{k'}$</td></tr>
<tr><td>$v_{k'} \geq D; v_k \geq D$</td><td>$D$</td></tr>
<tr><td>$q_k \geq Q$</td><td>Any value of D</td><td>0</td></tr>
<tr><td rowspan="5">Nonstrategic attacker is $n_{k'}$-type</td><td>$q_k \leq Q$</td><td>Any value of D</td><td>0</td></tr>
<tr><td rowspan="4">$q_k > Q$</td><td>$v_k + v_{k'} < D$</td><td>0</td></tr>
<tr><td>$v_{k'} < D; v_k < D; v_k + v_{k'} \geq D$</td><td>$\kappa_v(v_k + v_{k'} - D)$</td></tr>
<tr><td>$v_{k'} < D; v_k \geq D$</td><td>$\kappa_v v_{k'}$</td></tr>
<tr><td>$v_{k'} \geq D; v_k \geq D$</td><td>$\kappa_v D$</td></tr>
</table>

Notes. $Q = \frac{v_{k'}}{v_{k'}+v_{k'}}$; $\kappa_v = \frac{v_{k'}}{v_k}$

- [The impact of defender's budget] In general, when the defender's budget D increases (see column "condition on defender's budget" in Table 6.3 from bottom to top), VOT initially increases then decreases (last column in Table 6.3). The rationale behind this is similar to the VOR case, hence, further discussion is omitted.

- [The impact of λ] As opposed to VOR, λ has no impact on VOT. This is because when the true type is n, the defender does not take λ into account in her budget allocation decision under either the symmetric or asymmetric information scenario.
- Finally, VOT decreases (resp., increases) with Δv if a higher (resp., lower) proportion of n-type attackers prefer the target that is also the most valuable from the defender's perspective. Put differently, the defender gains more under symmetric information if she orders the targets differently from the n-type attacker.

6.3 Numerical Study: VOR vs. VOT

Note that from characterizations in Propositions 4 and 5, the characterization of both VOR and VOT depend on the true type and target preference, hence it is analytically quite cumbersome to compare VOR and VOT and explore the impact of different parameters on the relative comparison. Therefore, in this section, we first take a holistic approach to define the expected VOR and VOT and then provide an illustrative numerical study to compare VOT and VOR. Recall from §§6.1 and 6.2 that VOR and VOT depend on the true type of the attacker as well as on the preferred target of the n-type attacker. To make a unified comparison, we first aggregate all the cases with their respective probabilities and obtain the expected VOR and VOT. Let $\overline{\mathbb{V}}_R$ and $\overline{\mathbb{V}}_T$ denote the expected VOR and VOT, respectively, where $\overline{\mathbb{V}}_R$ and $\overline{\mathbb{V}}_T$ are computed as follows:

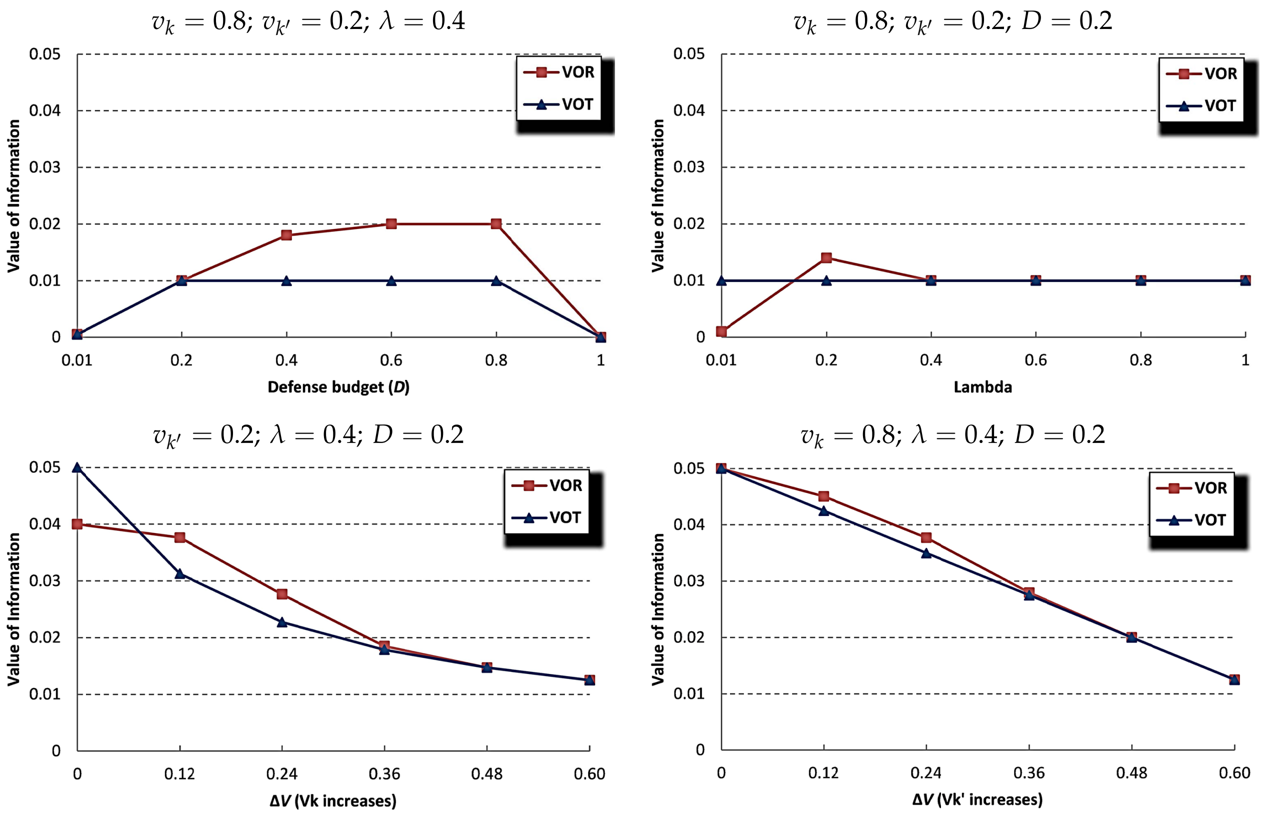
$$\overline{\mathbb{V}}_R = p_s[q_k \mathbb{V}_R^{s,k} + (1-q_k)\mathbb{V}_R^{s,k'}] + (1-p_s)[q_k \mathbb{V}_R^{n_k,k} + (1-q_k)\mathbb{V}_R^{n_{k'},k'}] \tag{6.1}$$

$$\overline{\mathbb{V}}_T = q_k[p_s \mathbb{V}_T^{s,k} + (1-p_s)\mathbb{V}_T^{n_k,k}] + (1-q_k)[p_s \mathbb{V}_T^{s,k'} + (1-p_s)\mathbb{V}_T^{n_{k'},k'}] \tag{6.2}$$

where $\mathbb{V}_T^{s,k}$ and $\mathbb{V}_T^{s,k'}$ are zero by definition. In Figure 6.1, keeping $p_s = q_k = 0.5$, we plot $\overline{\mathbb{V}}_R$ and $\overline{\mathbb{V}}_T$ with respect to D, λ and Δv. We make the following observations from the above comparisons:

- In general, from all of the above four figures, attacker's rationality information is more valuable than target information. In other words, lack of rationality information costs more to the defender than lack of target information does. The rationale behind this is as follows: as shown in the previous sections, the defender employs radically different defense strategies depending on whether the attacker is n- or s-type. Specifically, an n-type attacker requires for a more concentrated defensive effort, whereas an s-type attacker demands a more comprehensive defense strategy. This implies that using an s-type strategy against n-type attacker (or vice versa) imposes significant losses on the defender.

Figure 6.1: The comparison between VOR and VOT



- The gap between VOR and VOT initially increases in D. To understand this, consider a unit-dollar increase in the total budget. Faced with an s-type attacker, the defender distributes this extra dollar among all the defended targets in such a way that the expected damage is evenly reduced for all of them. On the other hand, in the case of an n-type attacker, the extra dollar is fully spent on the preferred target, as long as there is room for budget allocation for this target. Therefore, as D increases, the degree of mismatch between equilibrium budget allocations for s- and n-type attackers increases, which in turn makes the defense against an attacker with unknown degree of rationality more costly. In addition, if the defender knows the true degree of rationality of the attacker, under certain cases, she can use this to manipulate the attacker in his target selection (this is especially true if she knows the attacker to be s-type). Therefore, rationality information gives extra value to the defender, which reduces the impact of target information asymmetry, $\overline{\mathbb{V}}_T$ on the defender's payoff. To summarize, these two factors increase VOR at a faster rate than VOT as the defender has access to more resources.

- In general, the difference between VOR and VOT diminishes as λ increases. The main reason for this comes from the fact that the equilibrium budget allocation rule employed for the s-type attacker becomes more similar to that for n-type attacker as λ increases. Because of this, for high values of λ, the defender is equally worse off with both rationality and target information asymmetries.

- In order to explore which information is more valuable when targets become heterogeneous in terms of their values, we can either increase the value of target k while keeping k' constant (see the third panel in Figure 6.1), or decrease the value of target k' while keeping k constant (see the fourth panel in Figure 6.1). Note that the difference between VOR and VOT vanishes for high values of Δv, ($\Delta v = 0.6$). When Δv is small and both targets are low-valued (third panel of Figure 6.1), the defender may easily deter an s-type attack on both targets, causing her to be more concerned about the n-type's preferred target information. On the other hand, when Δv is small and both targets are high-valued (fourth panel of Figure 6.1), it is quite costly for the defender to protect these targets from an s-type attacker and consequently the defender is interested to know whether the attacker is s- or n-type.
- Finally, the difference between VOR and VOT may be affected by other values of p_s and q_k. In general, when the n-type attacker preference on targets is more likely to be similar to the s-type attacker (i.e., when q_k increases) then it becomes less valuable for the defender to know whether the attacker is s- or n-type. That is to say, the relative advantage of VOR over VOT decreases when q_k increases. However, the impact of change in p_s on the comparison between VOR and VOT mainly depends on whether or not the n-type attacker shares the same preference with the s-type attacker. Specifically, for low values of p_s, knowing the n-type attacker's preferred target becomes important for the defender when the n-type attacker prefers the less valuable target (target k'), under which the VOR may lose its relative advantage over VOT.

7 Extensions

Throughout the analysis of our basic model, we assume that s-type attacker shares the same target valuations with the defender. In §7.1, we relax this assumption. In addition, in asymmetric information analysis, we consider the cases, where either only rationality or target information is unknown by the defender. In §7.2, we analyze full asymmetric information case where both rationality and target information are unknown to the defender.

7.1 Uncertainty in s-type Attacker's Preference

In this section, we consider the case where s-type attacker's target valuations are not necessarily same as the defender's. For analytical tractability, we focus on the same two-target setting analyzed in §6. Namely, there are two targets k and k' with valuations v_k and $v_{k'}$, where $0 < v_{k'} \leq v_k \leq 1$. Similar to our approach before, using Harsanyi's transformation, we can define two types of s-type attacker: type 1 who values targets in the same way as the defender does; and, type 2 who values targets in the opposite way. Furthermore, we assume that q_k proportion is of type 1. Specifically, if u_{i_j} show the target i's valuation for type $j = 1, 2$ of s-type attacker, then we have $u_{k_1} = v_k$, $u_{k'_1} = v_{k'}$, $u_{k_2} = v_{k'}$, and $u_{k'_2} = v_k$. The rest of model has the same specifications as in the basic model of §4. The following proposition characterizes the defense equilibrium for this case.

Proposition 6. *Assume that the attacker is s-type and his preferred target is unknown for the defender. In a two-target setting, the defender's more valuable target, i.e., target k, should be always defended, whereas the defender uses a threshold-type policy to decide whether to defend the less valuable target, k'. Specifically, target k' is defended if and only if either $q_k < \frac{v_k v_{k'}}{v_k^2 + v_{k'}^2}$ and $D \geq \frac{\lambda v_{k'}(v_k - v_{k'})}{v_k}$, or, $q_k \geq \frac{v_k v_{k'}}{v_k^2 + v_{k'}^2}$ and $D > \lambda(v_k - v_{k'})$.*

There are two main take-aways from Proposition 6. First, consistent with our earlier results, defender uses a threshold-type policy to decide whether to defend a target or not. Second, the defender always sorts the targets in terms of their valuations and allocates budget to the more valuable target regardless of the s-type attacker's preferred target. This is in contrast with our finding in Proposition 3, where the defender, facing with a n-type attacker with unknown target information, sorts the targets with respect to belief-adjusted valuations (i.e., $q_k v_k$), and consequently may or may not defend the target with the highest valuation. This comes from the difference between s- and n-type behaviors in the target selection. Specifically, the defender cannot influence n-type attacker's target selection decision with her budget allocation decision, whereas she can incentivize s-type attacker to choose his second most valuable target through budget allocation decision. Therefore, the defender can be always better off by defending her most valuable target when she faces with s-type attacker.

7.2 Allocation Equilibrium under Full Asymmetric Information

The following proposition provides the defender's equilibrium strategy under full asymmetric information.

Proposition 7. *Assume that the attacker is s-type with probability p_s and n-type with probability $1 - p_s$. If the attacker is n-type, he prefers target k with probability q_k. Then, there exists a BNE in which*

1. *[Partition of targets] All the targets are partitioned into two disjoint sets denoted by I_1 and I_2 (as characterized by Algorithm 3 in Appendix), where $I_1 \cap I_2 = \varnothing$ and $I_1 \cup I_2 = \{1, ..., N\}$.*
2. *[Prioritization of targets] The defender prioritizes all the targets in I_1 and I_2 with respect to v_i and $q_i v_i$, respectively.*
3. *[Distribution of budget in I_1 and I_2] The defender distributes budget to the i^{th} most valuable target (in the sense of v_i) in I_1 if and only if $b_1^i \geq v_{k^*}$ and $v_i \geq b_2^i$, where k^* denotes the least valuable target (in the sense of*

$q_i v_i$) in subset I_2. The defender distributes budget to all targets in I_2. The thresholds for each target in subset I_1 are fully characterized in Algorithm 3 provided in Appendix.

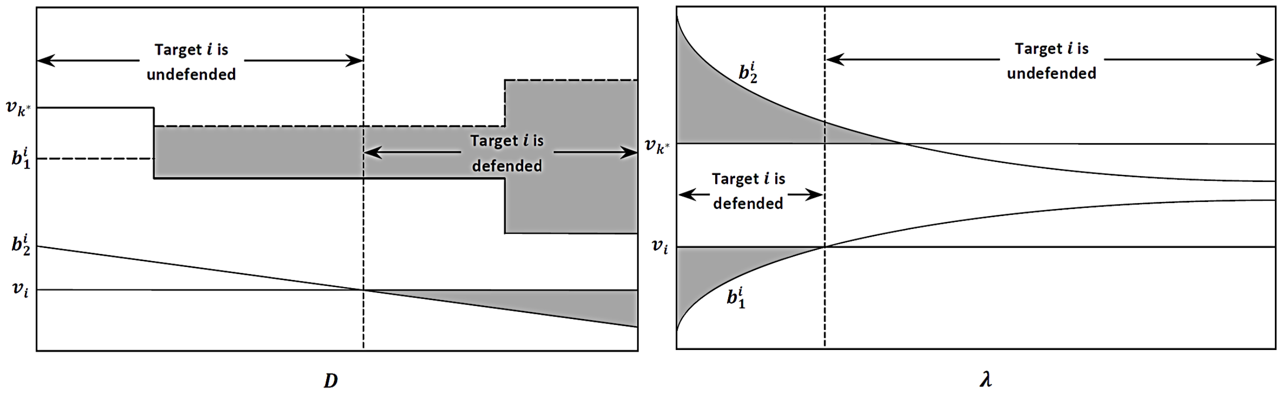
Note that the above proposition can be viewed as an extension of Propositions 2 and 3. Specifically, the defender divides the targets in two categories (i.e., $I_1$and I_2) and adopts the same prioritization and budget distribution schemes of Propositions 2 and 3 to defend the targets in first and second categories against s- and n-type attackers, respectively. Namely, the defender allocates budgets to the targets in I_1 to make sure that all the defended targets in I_1 would face equal expected damage from s-type attacker. On the other hand, to defend targets in I_2 against n-type attacker, the defender allocates the maximum available budget to each target in I_2, bounded by either target valuation v_i or the remaining budget (whichever is minimum). However, there is a subtle difference between the strategies used to defend in I_1 in the above Proposition and Proposition 2. Recall that in Proposition 2, when deciding whether to defend a specific target, say target i, the defender takes into account the valuation of not only that target but also another (critical) target, which is exogenously given by the preferred target of n-type attacker. In the above Proposition, the defender still employs two thresholds. However, both the critical target k^* and the threshold used for the critical target are endogenously determined by the partition scheme provided in Algorithm 3 of Appendix. To illustrate this, we provide a numerical example in Figure 7.1. Note that in this example, the critical target k^*, whose valuation needs to be checked in order to decide whether the i^{th} most valuable target in set I_1 is defended or not changes endogenously as defender's total budget D increases. This is because the size of I_2 increases with the inclusion of another target into the set I_2 as defender can allocate more budget, and the target included becomes the new critical target k^*. Using the algorithm in Proposition 7, we can compute two thresholds b_1^i and b_2^i and check whether each threshold is satisfied or not, i.e., $b_1^i \geq v_{k^*}$ or $v_i \geq b_2^i$. Thus, target i is defended whenever both shaded areas overlap on the horizontal axis. Figures 7.1a and 7.1b show the behavior of the thresholds b_1^i and b_2^i with respect to the total budget D and to λ, respectively. As shown in Figure 7.1a, target i is more likely to be defended in D and less likely to be defended in λ. The underlying rationale behind the latter observation is as follows. The defense investment becomes less cost-effective as λ increases. This causes the defender to concentrate her budget allocation on fewer targets in order to reduce the total expected damage from an s-type attacker. Hence, the total number of defended targets decreases.

8 Conclusion and Future Research Directions

This paper studied the impact of information asymmetry about the terrorist's various attributes on equilibrium defensive budget allocation decision. To address our research questions, we considered two critical

Figure 7.1: Optimal defensive thresholds as a function of D and λ under full asymmetric information

(a) Optimal threshold as a function of D (b) Optimal threshold as a function of λ



information that affects the government's decision: (i) the degree of rationality of the terrorist, and (ii) the terrorist's target preference. To answer research questions 1 (How should a defender prioritize multiple targets and allocate limited budget among them when faced with two types of asymmetric information (degree of rationality and target preference) about the attacker?), we fully characterized the equilibrium defense strategy under various information scenarios. We showed that, under both symmetric and asymmetric information scenarios, the defender first ranks the targets according to a scheme that depends on the type of information available to her. Next, she distributes the budget to the targets by using a set of thresholds starting from the most valuable targets according to ranking rule employed. Addressing research question 2 (What is the impact of partial information on the defender's equilibrium budget allocation strategy?), we compared both the target ranking and budget distribution schemes under symmetric and asymmetric information scenarios. Our analysis shows that the conditions under which a target is being defended involve more conditions under asymmetric information scenarios, especially when the government knows less about the terrorist's degree of rationality. Second, the targets are ranked according to their valuations if the attacker is strategic; otherwise, when the defender expects to face with non-strategic attacker with unknown target preference, she should adjust the ranking by using her a-priori beliefs. Finally, to address research question 3 (From the defender's perspective, what is the value of information regarding the attacker's degree of rationality and target preference? How does the value of information depend on problem parameters such as defender's budget, targets' valuations, and effectiveness of the defender's budget?), we compared how much additional value the defender would gain by using the rationality versus target preference information in her budgeting decision and explored how these comparisons are affected by the problem parameters. Our analysis shows that: (1) the value of information regarding either terrorist's rationality or target preference is nonzero only if the degree of information asymmetry is sufficiently high; (2) the value of information initially increases and then decreases in government's budget; (3) the value of information decreases (resp., increases) with the degree of heterogeneity between targets if the nonstrategic terrorist's preference highly (resp., weakly) matches with the government's preference; and, (4) the effectiveness ratio of attack has no impact on VOT information, but, the impact of that on VOR depends on the true type of the terrorist. We also provided two extensions. In the first one, the strategic terrorist's target valuations are not necessarily same as the government's, and, in the second extension, both rationality and target information are unknown to the government. Our findings showed that the structural properties of budget allocation equilibrium, specifically, the way to prioritize the targets and distribute the budget, still hold true. The model presented in this paper can be extended in various ways. For example, our model is a non-zero-sum two-stage game in which players make decisions sequentially. An interesting extension would be to consider simultaneous-move games with asymmetric information, known as Colonel Blotto games in the literature. Another extension would be to explore dual-asymmetric information scenarios, where both government and terrorist have incomplete information about each other. Lastly, we believe that the analysis of information asymmetry in the defender-attacker problems presents fruitful research opportunities, and hope that our model will fuel future research in this field.

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A Appendix

Proof. of Proposition 2. Recall that this Proposition considers the case where the defender has asymmetric information regarding the true type of attacker. In other words, the attacker is of s-type with probability p_s and n_k-type with probability $1 - p_s$. Note that this case can be modeled by letting $q_k = 1$ and $q_i = 0$, $i \neq k$ in Model II. The defender optimization problem can then be written as follows:

$$
\begin{aligned}
\min \quad & p_s z + (1 - p_s) v_k \left(1 - \tfrac{D_k}{\lambda w}\right) && \text{(A.1)} \\
\text{s.t.} \quad & v_i \left(1 - \tfrac{D_i}{\lambda v_i}\right) \leq z,\ i = 1, ..., N && \text{(A.2)} \\
& \textstyle\sum_{i=1}^{N} D_i \leq D && \text{(A.3)} \\
& D_i \leq v_i,\ i = 1, ..., N && \text{(A.4)} \\
& D_i \geq 0,\ i = 1, ..., N && \text{(A.5)}
\end{aligned}
$$

Note that the above A.1 is a linear programming model with respect to D_i and z. That means, the constraint A.2 is either binding or non-binding for each target $i = 1, ..., N$. We denote the subset of targets i (excluding target k) for which the constraint A.2 is binding and non-binding by I_D and I_{ND}, respectively. Note that solving the binding and non-binding constraints for D_i would lead to $D_i = \lambda(v_i - z)$, $\forall i \in I_D$, and $D_i = 0$, $i \in I_{ND}$. Substituting these values for D_i into the above linear program would yield the following optimization problem for the defender:

$$
\begin{aligned}
\min \quad & p_s z + (1 - p_s) v_k \left(1 - \tfrac{D_k}{\lambda w}\right) && \text{(A.6)} \\
\text{s.t.} \quad & v_i \leq z,\ i \in I_{ND},\ i \neq k && \text{(A.7)} \\
& v_k \left(1 - \tfrac{D_k}{\lambda v_k}\right) \leq z && \text{(A.8)} \\
& D_k \leq v_k && \text{(A.9)} \\
& \textstyle\sum_{i \in I_D}^{N} \lambda(v_i - z) + D_k \leq D && \text{(A.10)}
\end{aligned}
$$

Since D_k appears in the objective function with a negative sign, we are interested to find the maximum value of D_k subject to constraints A.8, A.9 and A.10. Because the upper bound on D_k implied by the constraint A.9 is greater than the lower bound on D_k implied by the constraint A.8, we can conclude that $D_k = \min\{v_k,\ D - \sum_{i \in I_D} \lambda(v_i - z)\}$. Plugging this value in A.6 and removing the fixed values from objective function, the defender's optimization problem can be reduced to either A.11 or A.14 depending on whether $D_k = v_k$ or $D_k = D - \sum_{i \in I_D} \lambda(v_i - z)$. Below, we analyze each case separately:

- If $D_k = v_k$ or, equivalently, $v_k \leq D - \sum_{i \in I_D} \lambda(v_i - z)$, then, the defender's optimization problem is expressed as follows:

$$\min \quad p_s z + (1-p_s)v_k\left(1-\frac{D_k}{\lambda w}\right) \tag{A.11}$$

$$\text{s.t.} \quad v_i \le z,\ i \in I_{ND},\ i \neq k \tag{A.12}$$

$$\textstyle\sum_{i\in I_D}^{N} \lambda(v_i - z) + v_k \le D \tag{A.13}$$

Clearly, all the constraints in A.12 can be reduced to a single constraint; $\max\limits_{i\in I_{ND}}\{v_i\} \le z$. Also note that the constraint A.13 models the availability of defensive budget, D. At the optimality, we can show that at least one of the two constraints, i.e., $\max\limits_{i\in I_{ND}}\{v_i\} \le z$ or $\sum_{i\in I_D}^{N} \lambda(v_i - z) + v_k \le D$ would be binding. That means, the defender in equilibrium would defend as many targets as possible subject to the budget constraint in A.13. As a result of this, the optimal budget allocation will be of a threshold-type policy; i.e., starting with the most valuable target, the defender would keep adding targets with lower valuations until either the constraint A.13 becomes binding or there is no target left undefended.

- If $D_k = D - \sum_{i\in I_D} \lambda(v_i - z)$ or, equivalently $v_k > D - \sum_{i\in I_D} \lambda(v_i - z)$, then the defender's optimization problem is expressed as follows:

$$\min \quad \left(p_s - \frac{(1-p_s)v_k|I_D|}{w}\right) z + \left(\frac{(1-p_s)v_k}{w}\right) \textstyle\sum_{i\in I_D} v_i \tag{A.14}$$

$$\text{s.t.} \quad v_i \le z,\ i \in I_{ND},\ i \neq k \tag{A.15}$$

$$\textstyle\sum_{i\in I_D}^{N} \lambda(v_i - z) + \lambda(v_k - z) \le D \tag{A.16}$$

Again, constraints A.15 can be reduced to only one constraint; $\max\limits_{i\in I_{ND}}\{v_i\} \le z$. Similar to the above case, we can show that at least one of the constraints A.15 and A.16 would be binding in the optimal solution. So, in equilibrium, the defender would defend as many targets as possible subject to constraint A.16. But, note that the first and second terms in the objective function in A.14 are decreasing and increasing in$|\ I_D\ |$, respectively. Combining increasing and decreasing cases, we can characterize the optimal policy as follows: Keep adding the targets with respect to their valuations to I_D and stop as soon as either no improvement occurs in the objective value or constraint A.16 becomes binding. Note that if D is sufficiently high, then all targets would be defended in equilibrium. We summarize above discussion in Algorithm 2.

Algorithm 2 (Defense equilibrium under asymmetric information about attacker's rationality):

1. Suppose that target k is the nonstrategic attacker's preferred target. Order the targets, excluding target k, in decreasing order of their values such that 1 and $N-1$ indicate the most and the least valuable targets, respectively. Initialize $j = 1$.

2. Let $b_1^j = p_s - (j)\frac{(1-p_s)v_k}{w}$, and $b_2^j = \frac{\lambda(\sum_{i=1}^{j} v_i + v_k) - D}{\lambda(j+1)}$.

3. If $b_1^j \geq 0$ and $b_2^j < v_{j+1}$ then check whether j is the least valuable target (i.e., $j = N-1$) or not. If $j = N-1$, then set $\mathscr{B}$ =0 and go to 6, otherwise, set $j = j+1$ and return to 2.

4. If $b_1^j < 0$, then set $\mathscr{B} = v_j$ and go to 6.

5. If $b_2^j \geq v_{j+1}$, then set $\mathscr{B} = \frac{\lambda(\sum_{i=1}^{j} v_i + v_k) - D}{\lambda(j+1)}$ and go to 6.

6. Optimal defense allocation is as follows: set $D_i = \lambda(v_i - \mathscr{B})$, $i \leq j$, $i \neq k$, $D_i = 0$, $i > j$, $i \neq k$, and $D_k = \min\{[D - \sum_{i=1}^{j}[\lambda(v_i - \mathscr{B})]^+]^+, v_k\}$.

By using the above algorithm iteratively from $j = 1$ to $j = N-1$, one can characterize the conditions under which target $j \neq k$ is defended as well as the optimal defense budget allocated to that target. Note that the optimal defense allocation mainly depends on the value of $\mathscr{B}$ in Step 6, which comes from Step 4 or 5, depending on whether $b_1^j < 0$ or $b_2^j \geq v_{j+1}$ (whichever case is satisfied first). Below, we discuss each scenario separately:

- If $b_1^j < 0$ is satisfied first: from Step 4, assume that k^* is the smallest index for which $b_1^{k^*} < 0$ while $b_2^{k^*} < v_{k^*+1}$. It implies that $\mathscr{B} = v_{k^*}$ in Step 6. Furthermore, we have $b_1^i \geq 0$ and $b_2^i < v_{i+1}$, $\forall i < k^*$. But, $v_{i+1} \leq v_i$ suggests that $b_2^i < v_i$, $\forall i < k^*$. So, from Step 6, the optimal defense allocation is $D_i = \lambda(v_i - v_{k^*})$, $i < k^*$, $i \neq k$, and $D_i = 0$, $i > k^*$, $i \neq k$, specifically, $D_{k^*} = 0$. Finally, because b_1^j is decreasing in j, we have $b_1^i < 0$, $\forall i > k^*$.

- If $b_2^j \geq v_{j+1}$ satisfied first: from Step 5, assume that k^* is the smallest index for which $b_2^{k^*} \geq v_{k^*+1}$ while $b_1^{k^*} \geq 0$. It implies that $\mathscr{B} = \frac{\lambda\left(\sum_{i=1}^{k^*} v_i + v_k\right) - D}{\lambda(k^*+1)}$ in Step 6. Furthermore, we have $b_1^i \geq 0$ and $b_2^i < v_{i+1}$, $\forall i < k^*$. But, $v_{i+1} \leq v_i$. Therefore, $b_2^i < v_i$, $\forall i < k^*$. Now, we show that $b_2^{k^*} \leq v_{k^*}$. Suppose that $b_2^{k^*} > v_{k^*}$. It concludes $b_2^{k^*-1} > v_{k^*}$, which contradicts with the latter result that $b_2^i < v_{i+1}$, $\forall i < k$. So, from step 6, it is clear that $D_i = \lambda(v_i - \mathscr{B})$, $i \leq k^*$, $i \neq k$, and $D_i = 0$, $i > k^*$, $i \neq k$. It is also easy to check that $v_i \leq b_2^i$, $\forall i > k^*$. Suppose that $v_i > b_2^i$, $\forall i > k^*$. It concludes $v_{k^*+1} > b_2^{k^*}$, which contradicts with the immediate assumption that $b_2^{k^*} \geq v_{k^*+1}$.

From the above algorithm, we can show that target i is defended if and only if $b_1^i \geq 0$ and $v_i > b_2^i$. Note that $b_1^i \geq 0$ is equivalent to $\frac{p_s w}{i(1-p_s)} \geq v_k$. If $t_1^i = \frac{p_s w}{i(1-p_s)}$ then $b_1^i \geq 0$ is equivalent to $t_1^i \geq v_k$. For notational consistency, we define $t_2^i = \frac{\lambda \sum_{j=1}^{i} v_j + \lambda v_k - D}{\lambda(i+1)}$. So, $v_i \geq b_2^i$ is equivalent to $v_i \geq t_2^i$.

Now, let $S = \{i \mid t_1^i \geq v_k,\, v_i \geq t_2^i\}$. From step 6, since $v_k > 0$, target k is defended if and only if $D - \sum_{i=1}^{j} \lambda(v_i - \mathscr{B}) > 0$. By substituting optimal value of $\mathscr{B}$, it is straightforward to conclude that target k is defended if and only if $v_k > \frac{\lambda \sum_{i \in S} v_i - D}{\lambda |S|}$. $\square$

Proof. of Proposition 3. This Proposition considers the case where attacker is of n-type, however, the defender does not know the true preferred target of the attacker. Note that this case can be modeled by letting $p_s = 0$ in Model II. The reduced problem is then a knapsack problem with a minimizing objective function. Let $k_1 = \arg\max_i \{q_i v_i\}$. We assign defensive budget to target k until either the constraint 3.17 or 3.18 becomes binding. If the constraint 3.18 becomes binding first, we can then set $k_2 = \arg\max_{i \neq k_1} \{q_i v_i\}$ and use the following iterative approach: In each step, if all the targets have unique $q_i v_i$ values, then the equilibrium budget allocation is $D_i = \min\{v_i, [D - \sum_{j=0}^{i-1} v_j]^+\}, \forall i$, or, equivalently, target i is defended iff $D - \sum_{j=0}^{i-1} v_j \geq 0$ or $v_i \geq t_i$ where $t_i = \sum_{j=0}^{i} v_j - D$. Now, assume that there are some targets with the same $q_i v_i$. Let k, $(k+1)$, ..., $(k+n)$ indicate these targets. If we still have leftover budget after allocating the budget to the targets $i < k$, then we can show that any distribution of remaining budget, i.e., $D - \sum_{i=0}^{k-1} v_i$, among targets k, $(k+1)$, ..., $(k+n)$ can be an equilibrium. That means, we will have multiple defense equilibria if $D - \sum_{i=0}^{k-1} v_i > 0$, whereas we have unique equilibrium if $D - \sum_{i=0}^{k-1} v_i \leq 0$. Specifically, if $D - \sum_{i=0}^{k-1} v_i > 0$, then, any combination of α_k, $\alpha_{(k+1)}$, ..., $\alpha_{(k+n)}$, $0 \leq \alpha_i \leq 1$, such that $\alpha_k + \alpha_{(k+1)} + ... + \alpha_{(k+n)} = 1$ would yield the same equilibrium, where the defender would allocate the following budgets to defend the targets: $D_i = v_i$, $i < k$ and $D_i = \min\{v_i, \alpha_i[D - \sum_{j=0}^{i-1} v_j]^+\}$, $k \leq i \leq k+n$, and $D_i = \min\{v_i, [D - \sum_{j=0}^{i-1} v_j]^+\}$, $i > k+n$. On the other hand, if $D - \sum_{i=0}^{k-1} v_i \leq 0$, then the unique defender's equilibrium is $D_i = \min\{v_i, [D - \sum_{j=0}^{i-1} v_j]^+\}$, $i < k$ and $D_i = 0$, $i \geq k$. $\square$

Proof. of Proposition 4. We need to compare the defender's objective in equilibrium under symmetric and asymmetric information. Table A.1 shows the optimal defense allocation under symmetric information. From assumption 1, $w > \max_{i=1,...,N}\{v_i\}/\lambda$. Since $v_k > v_{k'}$ by assumption, in order to simplify the proof, without loss of generality, we can reduce w so that $w = \frac{v_k}{\lambda} + \epsilon$, where ϵ is infinitesimal number. Table A.2 shows the defensive budget allocation when the defender has only partial information about the attacker's rationality. We can calculate the defender's expected loss due to asymmetric information by comparing the defender's objective in different regions in Table A.1 with corresponding regions in Table A.2. Regarding Equation 3.11, the defender's loss function in our two-target problem is:

$$p_s \sum_{i \in k, k'} v_i \left(1 - \frac{D_i}{\lambda v_i}\right) I_{i=i^*} + (1 - p_s) \sum_{i \in k, k'} q_i v_i \left(1 - \frac{D_i}{v_k}\right) \tag{A.17}$$

In what follows, we denote the value of information by γ. Below, we categorize all possibilities when there is uncertainty about p_s.

Table A.1: Optimal defense allocation under different symmetric information scenarios

Defensive budget	Attacker is n-type whose preferred target is k (Figure a)	Attacker is n-type whose preferred target is k' (Figure b)	Attacker is s-type (Figure c)		
			$a.\, \Delta v \geq \frac{D}{\lambda}$	$b.\, v_k + v_{k'} \geq \frac{D}{\lambda}$	$c.\, v_k + v_{k'} < \frac{D}{\lambda}$
D_k	$\min\{D,\, v_k\}$	0	$\min\{D,\, v_k\}$	$\frac{D+\lambda\Delta v}{2}$	λv_k
$D_{k'}$	0	$\min\{D,\, v_{k'}\}$	0	$\frac{D-\lambda\Delta v}{2}$	$\lambda v_{k'}$

Figure a

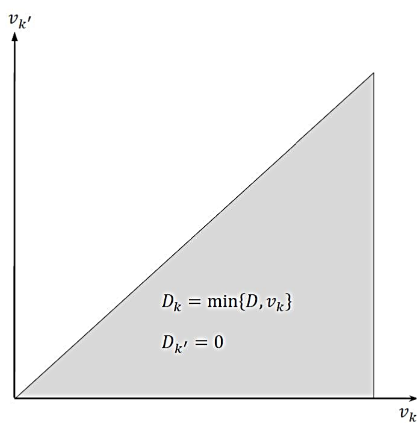


Figure b

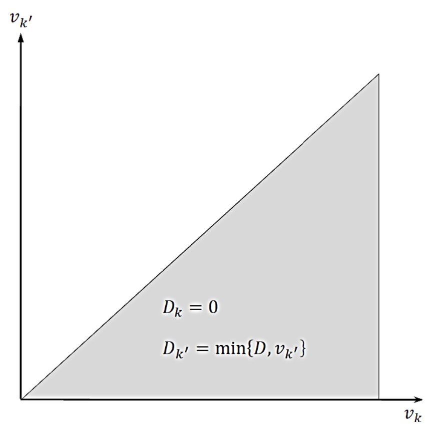
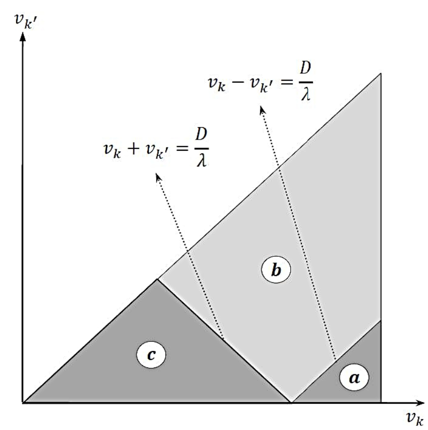


Figure c



- When the true type of the attacker is n-type and his preferred target is k (i.e., the most valuable target): We need to compare Figure a in Table A.1 with different regions of Figure a in Table A.2. From Equation A.17, under symmetric information, the defender's loss is equal to $v_k - \min\{D,\, v_k\}$. Since the defensive budget allocated to target k in regions 1, 2 and 5 (see Figure a in Table A.2) is equal to that of Figure a in Table A.1, we have $\gamma_1 = \gamma_2 = \gamma_5 = 0$. Let $p_s = 0$ and $q_k = 1$ in Equation A.17 which would yield $v_k - D_k$ as the defender's loss. Therefore, in regions 3 and 4, the value of information for the defender is $\min\{D,\, v_k\} - D_k$, which gives $\gamma_3 = \min\{D,\, v_k\} - \min\{D - \lambda v_{k'},\, v_k\}$, and $\gamma_4 = \min\{D,\, v_k\} - \frac{D+\lambda\Delta v}{2}$.

- When the true type of the attacker is n-type and his preferred target is k' (i.e., the least valuable target): We need to compare Figure b in Table A.1 with different regions of Figure b in Table A.2. From Equation A.17, under symmetric information, the defender's loss is equal to $v_{k'} - \kappa_v \min\{D,\, v_{k'}\}$. Therefore, the information has no value for the defender in regions 1 and 5 of Figure b in Table A.2, i.e., $\gamma_1 = \gamma_5 = 0$, since the budget allocated to target k' equals to that under symmetric case. Let $p_s = 0$ and $q_{k'} = 1$ in Equation A.17 which gives $v_{k'} - \kappa_v D_{k'}$ as defender's loss. The value of informational in regions 2, 3, and 4 is $\kappa_v[\min\{D,\, v_{k'}\} - D_{k'}]$, which gives $\gamma_2 = \kappa_v[\min\{D,\, v_{k'}\} - \min\{D - \lambda v_k,\, v_{k'}\}]$, $\gamma_3 = \kappa_v[\min\{D,\, v_{k'}\} - \frac{D-\lambda\Delta v}{2}]$, and $\gamma_4 = \kappa_v \min\{D,\, v_{k'}\}$.

Table A.2: Defender's budget equilibrium under partial information about p_s

n-type attacker's preferred target is k (Figure a)			
Region	Condition	D_k	$D_{k'}$
1	$p_s < \bar{t}$, $v_k + v_{k'} < \frac{D}{\lambda}$, $\Delta v < \frac{D}{\lambda}$	$\min\{D, v_k\}$	0
2	$p_s < \bar{t}$, $v_k + v_{k'} \geq \frac{D}{\lambda}$, $\Delta v < \frac{D}{\lambda}$	$\min\{D, v_k\}$	0
3	$p_s \geq \bar{t}$, $v_k + v_{k'} < \frac{D}{\lambda}$	$\min\{D - \lambda v_{k'}, v_k\}$	$\lambda v_{k'}$
4	$p_s \geq \bar{t}$, $v_k + v_{k'} \geq \frac{D}{\lambda}$, $\Delta v < \frac{D}{\lambda}$	$\frac{D+\lambda\Delta v}{2}$	$\frac{D-\lambda\Delta v}{2}$
5	$p_s \geq \bar{t}$, $\Delta v \geq \frac{D}{\lambda}$	$\min\{D, v_k\}$	0
n-type attacker's preferred target is k' (Figure b)			
Region	Condition	D_k	$D_{k'}$
1	$p_s < \underline{t}$, $v_k + v_{k'} < \frac{D}{\lambda}$	0	$\min\{D, v_{k'}\}$
2	$p_s \geq \underline{t}$, $v_k + v_{k'} < \frac{D}{\lambda}$	λv_k	$\min\{D - \lambda v_k, v_{k'}\}$
3	$p_s \geq \underline{t}$, $v_k + v_{k'} \geq \frac{D}{\lambda}$, $\Delta v < \frac{D}{\lambda}$	$\frac{D+\lambda\Delta v}{2}$	$\frac{D-\lambda\Delta v}{2}$
4	$p_s \geq \underline{t}$, $v_k + v_{k'} \geq \frac{D}{\lambda}$, $\Delta v \geq \frac{D}{\lambda}$	$\frac{D+\lambda\Delta v}{2}$	0
5	$p_s < \underline{t}$, $v_k + v_{k'} \geq \frac{D}{\lambda}$	0	$\min\{D, v_{k'}\}$

Figure a

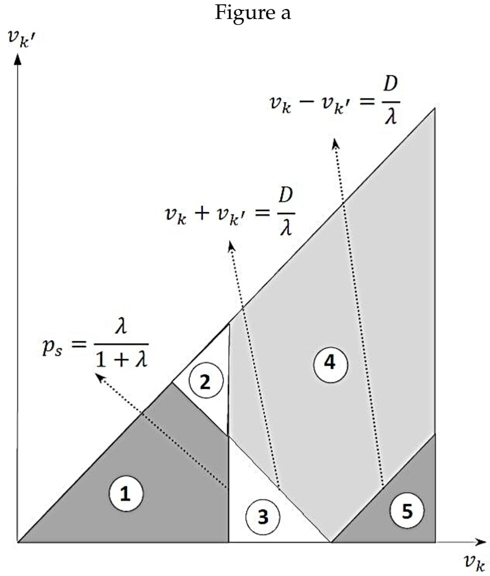
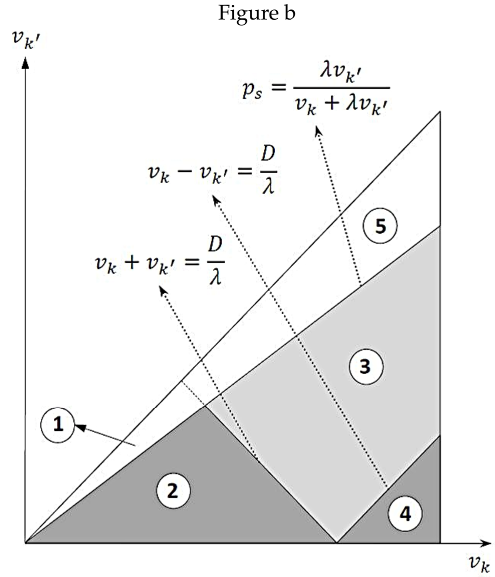


Figure b



Notes. $\bar{t} = \frac{\lambda}{1+\lambda}$; $\underline{t} = \frac{\lambda v_{k'}}{v_k + \lambda v_{k'}}$

- Attacker is s-type: The defender only has partial information about attacker's rationality, but she exactly knows the n-type attacker's preferred target. Let $p_s = 1$ and $I_{i=k} = 1$ in Equation A.17 whenever the strategic attacker attacks on target k, and $I_{i=k'} = 1$ whenever he strikes target k'. We have to consider two scenarios:

  1. The n-type attacker's preferred target is k: We need to compare different regions of Figure c in Table A.1 with corresponding regions of Figure a in Table A.2. It is clear that $\gamma_4 = \gamma_5 = 0$. Region 3 of Figure a in Table A.2 corresponds to region c of Figure c in Table A.1. In region 3 (Figure a in Table A.2), the defense on target k' is the same as what we have in region c

(Figure c in Table A.1), which is $\lambda v_{k'}$, and also the budget allocated to target k under asymmetric information is strictly greater than that under symmetric information. It readily means that $\gamma_3 = 0$. Note that, region 1 of Figure a in Table A.2 corresponds to region c of Figure c in Table A.1. In region c, the defensive budget allocated to both targets is big enough to deter any attack from both targets which means defender's loss is zero in that region. However, in Figure a (Table A.2), the budget allocated to target k is big enough to deter any attack by a strategic attacker. The strategic attacker therefore attacks on target k' and benefits the entire value of target k' which is $v_{k'}$. That is to say $\gamma_1 = v_{k'}$. Finally, region 2 of Figure a in Table A.2 corresponds to region b of Figure c in Table A.1. In region b, it is easy to verify that the defensive budget allocated to each target is not sufficient to deter an attack on that target. Therefore, the strategic attacker may strike target k or k' depending on problem parameters. We can show that, under symmetric information, the defender's loss is $v_k - \frac{D+\lambda\Delta v}{2\lambda}$ if the attacker strikes target k, and it is $v_{k'} - \frac{D-\lambda\Delta v}{2\lambda}$ if he strikes target k'. On the other hand, considering region 2 of Figure a in Table A.2, the attacker can benefit the whole value of target k', i.e., $v_{k'}$ if he attacks on k'. While $D_k = \min\{D, v_k\}$ in region 2 (Figure a in Table A.2), the s-type attacker is deterred from k and strikes target k' when $D \geq \lambda v_k$. Consequently, the value of information is $\gamma_2 = \frac{D-\lambda\Delta v}{2\lambda}$. If $\lambda\Delta v \leq D < \lambda v_k$, the s-type attacker strikes target k' and $\gamma_2 = \frac{D-\lambda\Delta v}{2\lambda}$.

2. The n-type attacker's preferred target is k': We need to compare different regions of Figure c in Table A.1 to their corresponding regions of Figure b in Table A.2. Since our approach is similar to part (1), we only summarize the results in Table A.3.

Table A.3: Value of attacker's rationality information

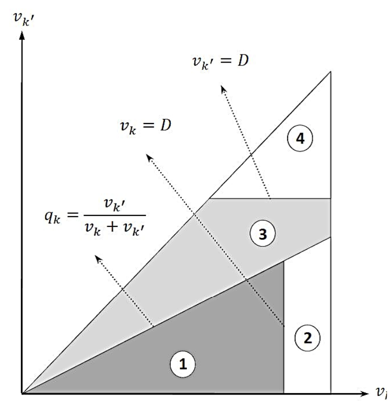
	Attacker's type			
Region	n-type whose preference is k	n-type whose preference is k	s- type attacker	
			n-type prefers k	n-type prefers k'
1	0	0	$v_{k'}$	v_k
2	0	$\kappa_v[\min\{D, v_{k'}\} - \min\{D - \lambda v_k, v_{k'}\}]$	$\frac{D-\lambda\Delta v}{2\lambda}$	0
3	$\min\{D, v_k\} - \min\{D - \lambda v_{k'}, v_k\}$	$\kappa_v \left[\min\{D, v_{k'}\} - \frac{D-\lambda\Delta v}{2}\right]$	0	0
4	$\min\{D, v_k\} - \frac{D+\lambda\Delta v}{2}$	$\kappa_v \min\{D, v_{k'}\}$	0	0
5	0	0	0	$\frac{D+\lambda\Delta v}{2\lambda}$

It is trivial to derive the results in Proposition 4 from Table A.2 and A.3. □

Proof. of Proposition 5. We need to compare the defender's loss under symmetric information when the attacker is n-type (Figures a and b in Table A.1) and defender's loss when she has only partial information about q (Table A.4). It leads us to consider two cases: attacker is n-type who prefers (i) the most valuable target (target k); and, (ii) the least valuable target (target k'). Note that, to obtain the defender's loss, we set $p_s = 0$ in Equation A.17.

Table A.4: Defender's defense equilibrium under partial information about q

Region	Condition	D_k	$D_{k'}$
1	$q_k > Q, D \geq v_k$	v_k	$\min\{v_{k'}, D - v_k\}$
2	$q_k > Q, D < v_k$	D	0
3	$q_k \leq Q, D \geq v_{k'}$	$\min\{v_k, D - v_{k'}\}$	$v_{k'}$
4	$q_k \leq Q, D < v_{k'}$	0	D



Note. $Q = \frac{v_{k'}}{v_k + v_{k'}}$

- When the true type of the attacker is n-type and his preferred target is k (i.e., the most valuable target): Let $q_k = 1$ in Equation A.17. We need to compare different regions of Table A.4 to Figure a in Table A.1. Note that, under symmetric information, only target k is defended (see Figure a in Table A.1), and the optimal defense budget that should be allocated to target k is $\min\{D, v_k\}$. Verify that this budget is equal to the budget that should be allocated to target k in regions 1 and 2 in Table A.4. This observation concludes $\gamma_1 = \gamma_2 = 0$. In region 3 in Table A.4, the defender allocates $D - v_{k'}$ to target k which is different to what she allocated to target k under symmetric information scenario. The defender's loss under symmetric information is $v_k - \min\{D, v_k\}$, while it is $v_k - \min\{v_k, D - v_{k'}\}$ under asymmetric information. That means $\gamma_3 = \min\{D, v_k\} - \min\{v_k, D - v_{k'}\}$. Finally, in region 4, the defender leaves target k undefended and the attacker benefits the entire value of target k under asymmetric information scenario. Since in region 4 we have $\min\{D, v_k\} = D$, the defender's expected loss under symmetric scenario is $v_k - D$. Therefore, we have $\gamma_4 = D$.
- When the true type of the attacker is n-type and his preferred target is k' (i.e., the least valuable target): Let $q_{k'} = 1$ in Equation A.17. We need to compare different regions of Table A.4 to Figure b in Table A.1 where only target k' is defended in equilibrium, and the optimal level of defense is $\min\{D, v_{k'}\}$. Note that, this budget is equal to the budget that should be allocated to target k' in regions 3 and 4 in Table A.4. That is to say, $\gamma_3 = \gamma_4 = 0$. In region 1, the defender allocated $D - v_k$ to target k' which is different to what she allocated to target k under symmetric information. The defender's loss under symmetric information is $v_{k'}(1 - \kappa_v)$, however it is $v_{k'} - \kappa_v(D - v_k)$ under asymmetric information. That means $\gamma_1 = \kappa_v(v_k + v_{k'} - D)$. Finally, in region 2, the defender leaves target k' undefended and the attacker obtains the whole value of target k' under asymmetric information. On

the other hand, the defender's loss under symmetric scenario is $v_{k'} - \kappa_v \min\{D, v_{k'}\}$. Therefore, we have $\gamma_2 = \kappa_v \min\{D, v_{k'}\}$.

We summarize our results in Table A.5. It is straightforward to derive the results in Proposition 5 from Table A.5.

Table A.5: Value of nonstrategic attacker's preference information

Region	n-type whose preference is k	n-type whose preference is k
1	0	$\kappa_v[\min\{D, v_{k'}\} - \min\{v_{k'}, D - v_k\}]$
2	0	$\kappa_v \min\{D, v_{k'}\}$
3	$\min\{D, v_k\} - \min\{v_k, D - v_{k'}\}$	0
4	D	0

□

Proof. of Proposition 6. The defender's optimization problem is:

$$\min \quad q_k \sum_{i=k}^{k'} v_i \left(1 - \frac{D_i}{\lambda u_{i_1}}\right) I_{i=i^*} + (1 - q_k) \sum_{i=k}^{k'} v_i \left(1 - \frac{D_i}{\lambda u_{i_2}}\right) J_{i=i^*} \tag{A.18}$$

$$\text{s.t.} \quad D_k + D_{k'} \le D \tag{A.19}$$

$$D_i \le v_i,\ i = k, k' \tag{A.20}$$

$$D_i \ge 0,\ i = k, k' \tag{A.21}$$

where $I_{i=i^*}$ and $J_{i=i^*}$ are the binary indicators for the s-type attacker whose preferred target is k and k', respectively, where $i^* = \arg\max\limits_{k,k'} \left\{u_{i_j} P(D_i, A_i^{s_j}) - A_i^{s_j}\right\}$, $j = 1, 2$. Similar to our earlier approach, we can remove binary indicators by introducing variables z_1 and z_2 that bound the maximum payoff for the s-type attacker:

$$\min \quad q_k z_1 + (1 - q_k) z_2 \tag{A.22}$$

$$\text{s.t.} \quad v_i \left(1 - \frac{D_i}{\lambda u_{i_1}}\right) \le z_1,\ i = k, k' \tag{A.23}$$

$$v_i \left(1 - \frac{D_i}{\lambda u_{i_2}}\right) \le z_2,\ i = k, k' \tag{A.24}$$

$$D_k + D_{k'} \le D \tag{A.25}$$

$$D_i \le v_i,\ i = k, k' \tag{A.26}$$

$$D_i \ge 0,\ i = k, k' \tag{A.27}$$

Note that the objective function of the above problem is a knapsack problem in terms of z_1 and z_2. If

$q_k \geq 0.5$ then the defender would minimize z_1 as much as possible. Since z_1 corresponds to the s-type attacker whose preferred target is k, decreasing z_1 corresponds to the defender's defending target k as per constraint A.23 for $i = k$. By assigning defense to target k, i.e., increasing D_k, both z_1 and z_2 decrease according to constraints A.23 and A.24. However, the rate of decrease in z_2 is greater than that of z_1 because $| \frac{\partial z_2}{\partial D_k} | \geq | \frac{\partial z_1}{\partial D_k} |$. While $D_{k'} = 0$, the defender can increase D_k until one of the constraints A.24 (for $i = k'$) or A.25 becomes binding. Specifically, if A.25 becomes binding first, the optimal solution is $D_k = D$ and $D_{k'} = 0$, otherwise if constraint A.24 becomes binding, then the solution would yield $z_2 = v_{k'}$. In the latter case, we can obtain the value of D_k by solving the constraint A.24 for D_k when $i = k$: $D_k = \frac{\lambda v_{k'}(v_k - v_{k'})}{v_k}$. Since constraint A.25 is not binding, we still have some leftover budget. Then, we can continue optimizing objective function by decreasing z_1 until either constraint A.25 or A.23 (for $i = k'$) becomes binding. In the first case, the optimal solution is $D_k = D$ and $D_{k'} = 0$. However, in the latter case, the optimal solution would yield $D_k = \lambda(v_k - v_{k'})$. By comparing these two cases, we can show that the constraint A.25 cannot be binding when $D \geq \lambda(v_k - v_{k'})$. This in turn implies that the defender can be better off by decreasing z_2, i.e., defending target k'. To summarize, we can conclude that target k' is defended when D is sufficiently large, i.e., $D \geq \lambda(v_k - v_{k'})$.

Now assume that $q_k < 0.5$. The defender would then minimize z_2 as much as possible. Note that, from constraints A.24, the rate of decrease in z_2 is higher when the defender starts defending target k than when she defends target k', i.e., $| \frac{\partial z_2}{\partial D_k} | \geq | \frac{\partial z_2}{\partial D_{k'}} |$. By allocating budget to target k, both z_1 and z_2 decrease according to constraints A.23 and A.24 (for $i = k$). Because the rate of decrease in z_2 is greater than that of z_1, the defender can increase D_k until one of the following three conditions happens: (i) constraint A.25 becomes binding; (ii) the objective function can be improved by decreasing z_1, which happens when $z_2 = \frac{q_k}{1-q_k} z_1$ or $D_k = \frac{\lambda(1-2q_k)v_k v_{k'}}{(1-q_k)v_k - q_k v_{k'}}$; (iii) the objective function can be improved by decreasing z_2 by defending target k'. Assuming that constraint A.25 is not binding, condition (iii) is satisfied before the condition (ii) when $q_k \leq \frac{v_k v_{k'}}{v_k^2 + v_{k'}^2}$, which is satisfied when constraint A.23 (for $i = k'$) becomes binding before $z_2 = \frac{q_k}{1-q_k} z_1$ holds. So, (i) is not satisfied when $D \geq \frac{\lambda v_{k'}(v_k - v_{k'})}{v_k}$. If $\frac{v_k v_{k'}}{v_k^2 + v_{k'}^2} < q_k \leq 0.5$, then condition (ii) holds before condition (iii). Constraint A.25 is then not the binding constraint when $D \geq \frac{\lambda(1-2q_k)v_k v_{k'}}{(1-q_k)v_k - q_k v_{k'}}$. In this case, the defender keeps decreasing z_1 by increasing defense on target k until one of constraints either A.24 (for $i = k'$) or A.25 becomes binding. Similar to the above discussion where $q_k \geq 0.5$, we can show that target k' is defended when $D \geq \lambda(v_k - v_{k'})$. □

Proof. of Proposition 7. From Model II, when defender is fully uninformed about both the attacker's type and the n-type attacker's preferred target, after removing the fixed terms from objective function, the defender's optimization problem can be written as follows:

$$\min \quad p_s z - (1-p_s)\sum_{i=1}^{N} \frac{q_i v_i D_i}{\lambda w} \tag{A.28}$$

$$\text{s.t.} \quad v_i\left(1-\frac{D_i}{\lambda v_i}\right) \leq z,\ i=1, ..., N \tag{A.29}$$

$$\sum_{i=1}^{N} D_i \leq D \tag{A.30}$$

$$D_i \leq v_i,\ i=1, ..., N \tag{A.31}$$

$$D_i \geq 0,\ i=1, ..., N \tag{A.32}$$

If I_D shows the set of targets for which constraint A.29 is binding, then $D_i = \lambda(v_i - z)$, $\forall i \in I_D$. Let I_{ND} shows the remaining targets. The defender's optimization problem can be then reduced to:

$$\min \quad \left(p_s + (1-p_s)\sum_{i\in I_D}\frac{q_i v_i}{w}\right) z - (1-p_s)\left(\sum_{i\in I_D}\frac{q_i v_i^2}{w} + \sum_{i\in I_{ND}}\frac{q_i v_i D_i}{\lambda w}\right) \tag{A.33}$$

$$\text{s.t.} \quad v_i\left(1-\frac{D_i}{\lambda v_i}\right) \leq z,\ i \in I_{ND} \tag{A.34}$$

$$\sum_{i\in I_D}\lambda(v_i - z) + \sum_{i\in I_{ND}} D_i \leq D \tag{A.35}$$

$$D_i \leq v_i,\ i \in I_{ND} \tag{A.36}$$

$$D_i \geq 0,\ i \in I_{ND} \tag{A.37}$$

It is clear that A.33 is a knapsack problem with respect to D_i, $i \in I_{ND}$ with minimization objective function. Since the coefficients in constraint A.35 are the same for all $i \in I_{ND}$, the optimal solution is to allocate the remaining defensive resources, i.e., $D - \sum_{i\in I_D}\lambda(v_i - z)$, to target j_1 as much as possible, where $j_1 = \arg\max_{i\in I_{ND}}\{q_i v_i\}$, until one of constraints A.35 or A.36 becomes binding. If A.36 first turns into a binding constraint, then, regarding the optimal solution of knapsack problem, the remaining defensive budget, i.e., $D - \sum_{i\in I_D}\lambda(v_i - z) - v_{j_1}$, should be allocated to target j_2, where $j_1 = \arg\max_{i\in\{I_{ND}-j_1\}}\{q_i v_i\}$. We continue this approach until either constraint A.35 becomes binding, i.e., D is entirely allocated, or all targets $i \in I_{ND}$ are defended. Let us define I_{ND_1} as the targets for which the relevant constraint A.36 becomes binding, i.e., $D_i = v_i$, $i \in I_{ND_1}$. After some simplifications, it is easy to verify that the defender's optimization problem can be reduced to:

$$\min \quad \left(p_s - (1-p_s)\frac{q_k v_k |I_D|}{w}\right) z + (1-p_s)\sum_{i\in I_D}\frac{q_i v_i (z - v_i)}{w} + (1-p_s)\frac{q_k v_k \sum_{i\in I_D} v_i}{w} + \mathscr{F}_{I_{ND_1}} \tag{A.38}$$

$$\text{s.t.} \quad v_i \leq z,\ i \in I_{ND} \tag{A.39}$$

$$\lambda\sum_{i\in I_D} v_i + \sum_{i\in I_{ND_1}} v_i - \lambda \mid I_D \mid z \leq D \tag{A.40}$$

where $\mathscr{F}_{I_{ND_1}}$ is a function of $\sum_{i\in I_{ND_1}} v_i$. Since the targets are ordered with respect to their valuations, we can reduce all constraints in A.39 to only one constraint; $\max_{i \in I_{ND}} \{v_i\} \leq z$. Note that the first part of the objective function is decreasing function in I_D, the second part is always negative (because $z < v_i$, $i \in I_D$), and the third part is increasing function in I_D. Together, this implies that, we have to keep adding the targets with respect to their valuations to I_D and stop as soon as either no improvement occurs in the objective value or constraint A.40 becomes binding. We summarize our proposed approach in Algorithm 3 to characterize the defender's equilibrium.

Algorithm 3 (Defense equilibrium under full asymmetric information):

1. Order the targets with respect to $q_i v_i$, such that j_1 indicates the target with the highest $q_i v_i$ and j_N shows the target with the lowest $q_i v_i$. Set $v_{j_0} = 0$ and $k = 1$. Moreover, partition all targets to two disjoint subsets $I_1 = \{1, 2, ..., N\}$ and $I_2 = \emptyset$, where targets in I_1 are prioritized in decreasing order with respect to their valuations.
2. Pick target j_k from set I_1 and put it into set I_2. Reorder all the remaining targets in I_1 with respect to their valuations, such that 1 indicates the target with the highest valuation and $N - k$ shows the target with the lowest valuation. Set $j = 1$.
3. Let $b_1^j = p_s + (1 - p_s)\left(\sum_{i=1}^{j} \frac{q_i v_i}{w} - (j)\frac{q_{j_k} v_{j_k}}{w}\right)$, $b_2^j = \frac{\lambda\left(\sum_{i=1}^{j} v_i + v_{j_k}\right) + \sum_{t=0}^{k-1} v_{j_t} - D}{\lambda(j+1)}$.
4. If $b_1^j \geq 0$, and $b_2^j < v_{j+1}$, then check whether j is the least valuable target in set I_1 or not. If $j < N - k$, i.e., j is not the least valuable target, then let $j = j + 1$ and return to 3, however, if j is the least valuable target, i.e., $j = N - k$, then $\mathscr{B} = 0$ and go to 7.
5. If $b_1^j < 0$, then $\mathscr{B} = v_j$ and go to 7.
6. If $b_2^j \geq v_{j+1}$, then $\mathscr{B} = \frac{\lambda\left(\sum_{i=1}^{j} v_i + v_{j_k}\right) + \sum_{t=0}^{k-1} v_{j_t} - D}{\lambda(j+1)}$ and go to 7.
7. If $D - \sum_{i=1}^{j} \lambda(v_i - \mathscr{B}) - \sum_{t=0}^{k-1} v_{j_t} > v_{j_k}$, then let $k = k + 1$ and return to 2. Otherwise, go to 8.
8. Optimal defense allocation is as follows: set $D_i = \lambda(v_i - \mathscr{B})$, $i \in I_1$, $i \leq j$, $D_i = 0$, $, i \in I_1$, $i > j$, $D_{j_t} = v_{j_t}$, $j_t \in I_2$, $t < k$, and $D_{j_k} = D - \sum_{i=1}^{j} \lambda(v_i - \mathscr{B}) - \sum_{t=0}^{k-1} v_{j_t}$.

By iterating the above algorithm for all targets, we can show that under fully incomplete information, only a combination of the most valuable targets and the nonstrategic attacker's most preferred targets will receive the defensive resources. In particular, the nonstrategic attacker's most preferred targets are fully defended, i.e., $D_i = v_i$, $i \in I_2$, and among the remainders, only a set of most valuable targets are defended. Specifically, if k^* indicates the smallest k for which one goes from step 7 to step 8, it is easy to show that target $i \notin I_2$ is defended if and only if $b_1^i \geq v_{k^*}$, and $v_i > b_2^i$, where: $b_1^i = \frac{p_s w}{i(1-p_s)q_{k^*}} + \sum_{j=1}^{i} \frac{q_j v_j}{i q_{k^*}}$, and $b_2^i = \frac{\lambda(\sum_{j=1, j \neq k^*}^{i} v_j + v_{k^*}) + \sum_{j \in F} v_j - D}{\lambda(i+1)}$. $\square$