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The impact of product substitution and retail capacity on the timing and depth of price promotions: theory and evidence

Mehmet Gümüş[a*], Philip Kaminsky[b] and Sameer Mathur[c]

[a]*Desautels Faculty of Management, McGill University, Montréal, Canada;* [b]*Department of Industrial Engineering and Operations Research, University of California at Berkeley, Berkeley, CA, USA;* [c]*Marketing Area, Indian Institute of Management, Lucknow, India*

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We investigate the impact of store capacity and extent of inter-product substitution in a retailer's assortment on the optimal timing and depth of price promotions. We develop a stylised model of a monopolistic retailer selling two substitutable products over time, where demand for each product in each period is a function of the prices of both products in that and earlier periods as well as the degree of substitution between the two periods. We present closed-form solutions to limiting cases of the model, and observe the following: When retailers optimise profits, (1) price promotions are relatively deeper in both absolute and relative terms at higher capacity stores than at low capacity stores, (2) price promotions for more expensive products are relatively deeper (shallower) in both absolute and relative terms than price promotions for cheaper products if the degree of substitution is low (high) and (3) the products are sequentially promoted if the degree of substitution is low, and simultaneously promoted if the degree of substitution is high. To confirm that these insights from a simple stylised two-product model are relevant in practice, we survey price promotions within the shampoo and detergent assortments of four mass-market retailers, and observe behaviour corresponding to the results from our stylised model.

Keywords: promotions; pricing; retailing

1. Introduction

Effective revenue management strategies for fast moving consumer goods often focus on price promotions. Indeed, price promotions account for 31% of marketing budgets on average (Gelb, Andrews, and Lam 2007). In settings with competing products, however, determining the optimal depth and timing of sales is complicated by potential inter-product and inter-temporal substitutions. Intuitively, issues of consumer loyalty and price sensitivity interact with operational issues like inventory or stocking levels of competing products. All of these should have a role to play in determining effective promotion plans, although the precise nature of that role is not immediately apparent. In this paper, we explore these issues focusing on how the interplay of inter-temporal and inter-product substitution, price sensitivity and available store capacity (shelf space, or maximum stocking level) impacts the timing and depth of price promotions.

Consider, for example, the shampoo assortment carried at two representative stores of large Canadian retailer chains, $Metro$ and $Provigo$ (note that these are two of the four chains represented in a data-set that will be discussed later in this document). The stores vary in terms of the store capacity devoted to shampoo – $Metro$ devotes 8.5 m^2 of aisle space to stocking its shampoo assortment, while $Provigo$, being relatively larger, devotes 33.8 m^2 of aisle space to stocking shampoos. $Metro$ and $Provigo$ also vary in the extent of potential inter-product substitution opportunities – $Metro$ offers 44 unique shampoos, while $Provigo$ offers 77, suggesting a higher degree of potential substitution at $Provigo$. In addition, both retailers offer both inexpensive and more expensive shampoos within their assortments (for example, $Provigo$ sells equal-sized bottles of $Sunsilk\ Thermashine$ for \$3.95 and $Dove\ Go\ Fresh$ for \$6.95.) Together, these retail characteristics raise a variety of challenging questions. For example, should $Metro$ set relatively deeper or shallower price discounts on its shampoos, compared to $Provigo$, which has larger capacity than $Metro$? Should price promotions be relatively larger for more expensive (e.g. $Dove$) or inexpensive (e.g. $Sunsilk$) products? Does the degree of inter-product substitution (potentially high at $Provigo$, lower at $Metro$) influence the depth of price promotions? Should $Metro$ and $Provigo$ promote $Dove$ and $Sunsilk$ shampoos simultaneously, or should they be promoted sequentially? Does the degree of inter-product substitution in their assortment (potentially high at $Provigo$, low at $Metro$) influence the timing of price promotions?

Indeed, the marketing literature provides compelling empirical evidence for the impact of inter-temporal and inter-product substitution effects on demand. Draganska and Jain (2006) provide evidence of inter-product demand interactions and show that firms exploit these interactions to differentiate between customers with differing valuations. Pesendorfer (2002) analyses

*Corresponding author. Email: mehmet.gumus@mcgill.ca

price and sales data in a dynamic multi-product environment, and shows that demand for a particular product depends on both current and past prices of that product and its substitutes. For example, Pesendorfer (2002) shows that, holding other variables constant, a one per cent increase in last week's price for product (Heinz ketchup) increases current-period demand for product A by 2.1 per cent, and a one per cent increase in last week's price of product B (Hunts ketchup) increases current-period demand for A by one per cent. These findings suggest that, depending on pricing decisions, customers may delay their purchases – *wait for the sale* – or change their preferences – *switch products.*

There is also significant empirical evidence that promotion and stocking decisions need to be coordinated. Corsten and Gruen (2003), and Taylor and Fawcett (2001), for instance, show that out-of-stock rates for promoted items are significantly higher than those for non-promoted items.[1] They also analyse the root causes of out-of-stock incidences, and identify lack of coordination between pricing and stocking decisions as the most significant source of the problem.

1.1 *Research questions and summary of results*

In this paper, we address the following three research questions:

(1) How should the space available for a product in a store (the capacity) influence the *depth* of the discount offered during a price promotion?
(2) How should the potential for inter-product substitution within retail assortment influence the *depth* of discounts offered during price promotions, and should this effect be different for relatively cheap and relatively expensive products?
(3) How should inter-product substitution within retail assortment influence the *timing* of price promotions for both more expensive and inexpensive products?

To generate insight into these questions, we begin with a stylised discrete-time model of a monopolistic retailer selling two substitutable products over a finite number of time periods to customers who are heterogenous in several dimensions. In each period, the firm coordinates pricing and inventory-ordering decisions for the products. Demand for each product in each period is a function of the prices of both products in that period as well as in the previous period. Although even a relatively simple demand function with these characteristics leads to an intractable model, we are able to characterise optimal policies for several limiting cases, and observe the following: First, our analysis suggests that the optimal depth of price promotions for a product is increasing in the retail capacity allocated to that product. Second, we find that as the level of product substitution increases, the retailer should offer deeper and deeper discounts for the cheaper product relative to the discount offered for the more expensive product. Finally, we find that if the level of product substitution is relatively high, the retailer should promote products simultaneously, whereas as the degree of substitution decreases, the retailer benefits more from promoting products sequentially.

Our model is highly stylised, and also we analyse limiting cases of the model, so we turn to empirical analysis to confirm our observations. Specifically, we explore price promotion strategies within the shampoo and detergent categories at four mass-market retailers (Provigo, Metro, Loblaws, and IGA) located in Canada. Given the contents of our data-set, we assume that the number of unique products in a category carried by a store is a proxy for the degree of substitution in a category and the extent of aisle space allotted to a product category is a proxy for available store capacity for that category. Based on these assumptions, we make the following observations, which corroborate with our analytical results: Stores with *larger* capacity to stock products offer *deeper* discounts compared to stores with comparatively smaller retail capacity. Also, price promotions offered on more expensive products are deeper than those offered on cheaper products if the extent of inter-product substitution is *low*. Finally, stores *sequentially* promote products within the same category if the extent of inter-product substitution available within their assortment is *low*, and simultaneously promote products if the degree of substitution is high. To the best of our knowledge, our paper is the first to explore how store capacity and product substitution within retail assortment should correlate with the timing and depth of price promotions offered by retailers.

1.2 *Related literature*

There is a rich literature in operations management, often referred to as *revenue management*, that considers (i) time-based and (ii) product-based substitution effects on dynamic pricing problems. For instance, Ahn, Gümüş, and Kaminsky (2007) and Ahn, Gümüş, and Kaminsky (2009) consider a monopolistic firm selling a single product and analyse the impact of inter-temporal demand interactions on pricing and inventory-ordering decisions. Aviv and Pazgal (2008) and Elmaghraby, Gülcü, and Keskinocak (2008) characterise optimal markdown pricing policies with two or more price levels. Research in the second category models the impact of demand interactions among products on pricing decisions. In this stream of literature, a seller either makes assortment decisions without changing prices (see, e.g. Cachon and Kok 2007; Aydin and Hausman 2009) or as in our model, a seller dynamically changes the prices for a fixed assortment (see, e.g. Kimes 1989; Harris and Pinder 1995; Aydin and Porteus 2008; Dong, Kouvelis, and Tian 2009). Review papers by McGill and van Ryzin (1999), Elmaghraby and Keskinocak (2003) and Shen and Su (2007) provide comprehensive surveys of this literature. To the best of our knowledge, the models in this literature focus on either inter-temporal demand substitution in a single-product framework or inter-product demand substitution in a single/multi-period setting with no interactions between periods. Our analysis enriches this literature by modelling both inter-temporal and inter-product substitution effects in a capacitated retail environment, and analysing the impact of capacity and substitution on promotion decisions. In addition to revenue management where inventory is not replenished after the starting of the selling season, researchers also consider joint dynamic pricing and inventory control under variety of assumptions, including no set-up cost (Federgruen and Heching 1999), non-zero set-up cost (Chen and Simchi-Levi 2004; Huh and Janakiraman 2008), lost sales (Chen, Ray, and Song 2006) and multiplicative demand uncertainty (Song, Ray, and Boyaci 2009). A comprehensive survey of joint dynamic pricing and inventory control can be found in Chan et al. (2004). However, most of these models employ simple demand functions in which demand is affected only by current pricing decisions.

There is also significant prior research on price promotions in the marketing literature, including theoretical and empirical studies as well as studies combining theory and empirical analysis. *Theoretical* articles on price promotions in the marketing literature have dealt with diverse issues such as firm and consumer asymmetry (Narasimhan 1988), dynamic competition (Lal 1990a), interaction between manufacturers and retailers (Lal and Villas-Boas 1998), and price-matching guarantees (Chen, Narasimhan, and John Zhang 2001). *Empirical* articles on price promotions in the marketing literature have tested the implications of inter-temporal price dispersion models developed in Varian (1980) and Narasimhan (1988). In these models, the temporary price promotions are interpreted as mixed-strategy equilibria and caused by customer search costs. Extensive literature reviews are also available (see Villas-Boas 1995; Rao, Arjunji, and Murthi 1995; Blattberg, Briesch, and Fox 1995). However, we are aware of little effort to investigate how retail capacity and degree of product substitution in retail assortment influence the timing and depth of price promotions. Our paper attempts to fill this gap in the literature.

Past research related to our paper includes (Lal 1990a; Freimer and Horsky 2008; Silva-Risso, Bucklin, and Morrison 1999). Lal (1990a) models the timing of price promotions by two national firms and one local firm competing in a market with switchers and loyal customers. He shows that the national firms should set price promotions sequentially, if there are a sufficiently large number of switchers in the market. Our paper enriches this line of thinking by analysing how the degree of product substitution in a retailer's assortment drives the timing of price promotions. Freimer and Horsky (2008) explain the effect of consumer learning on price promotions. They examine whether it is optimal for competing national brands to offer periodic price promotions when price changes allow consumers to learn by trying out a product. They find that brands periodically reduce their prices to induce purchases by non-buyers of the brands. These consumers may, after learning about the characteristics and fit of the brands, decide to continue purchasing them even if their prices are later increased. In related work, (Silva-Risso, Bucklin, and Morrison 1999) propose a decision-support system for an optimal sales promotion calendar for the frequency, depth and duration of deals and how these are affected by changes in market response, competitive activities and pass-through rates. Our paper enriches this literature by studying how retail assortment characteristics impact the optimal timing and depth of price promotions.

2. Model and analysis

To generate initial insight into the research questions outlined above, we develop a stylised model of a monopolistic retailer with heterogeneous customers.

2.1 *Retailer*

Consider a monopolistic retailer who sells two substitutable products belonging to a common product category over a finite number of time periods. Let $\boldsymbol{M} = \{1, 2\}$ denote the set of product indices and let $\boldsymbol{T} = \{1, \ldots, T\}$ denote the set of time indices. $c^i \in \boldsymbol{M}$ denotes the retailer's unit cost of product i, and without loss of generality, let $c^1 \leq c^2$, so that product 1 is relatively cheaper than product 2. At the beginning of each period $t \in \boldsymbol{T}$, the retailer sets the unit prices, p_t^i, for both products $i \in \boldsymbol{M}$, where prices are scaled to range between 0 and 1, so $0 \leq p_t^i \leq 1$. Finally, let q denote the maximum store capacity, and let $D_t^i(\boldsymbol{p})$ denote the demand for product i at period t, where $\boldsymbol{p}$ represents the vector of prices across both products and all time periods.

Specifically, to address the research questions raised in Section 1, we develop a simple, stylised demand function D_t^i that has three components, one representing the demand for each product in that period due to the price of that product (the standard linear demand function, which we call *base* demand), one representing inter-temporal demand, which is specific to each product (the demand in this period because the price for a product is lower than the price of that product in the

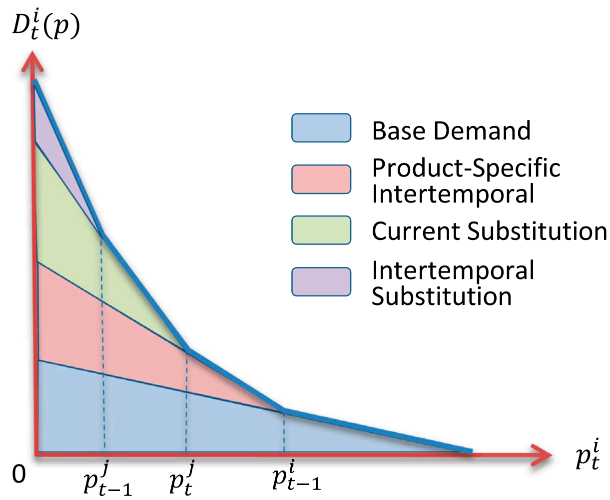


Figure 1. Illustration of demand function (assuming $p_{t-1}^j \leq p_t^j \leq p_{t-1}^i$).

previous period) and one representing inter-product demand due to demand that was originally for other products, both in this period and in the previous period (substitution). We utilise the following function, which captures these characteristics in a reasonable but relatively tractable way, and discuss this function in more detail below:

$$D_t^i(\boldsymbol{p}) = \overbrace{1 - p_t^i}^{\text{Base Demand}} + (1-\beta)\,\alpha\,\overbrace{\left[p_{t-1}^i - p_t^i\right]^+}^{\substack{\text{Product Specific}\\ \text{Intertemporal Effect}}} + \beta \overbrace{\left[\overbrace{\left[p_t^j - p_t^i\right]^+}^{\text{Current Substitution}} + \alpha\,\overbrace{\left[\min(p_{t-1}^i, p_{t-1}^j) - p_t^i\right]^+}^{\text{Intertemporal Substitution}}\right]}^{\text{Substitution Effects}}$$

where $[x]^+ = \max(x, 0)$, $i, j \in \{1, 2\}$ and $i \neq j$.

The three components of the demand function are labelled in the equation above and illustrated in Figure 1. *Base demand* for a product i represents the demand for product i in period t (as a fraction of the total potential period t market) that will result from pricing the product at p_t^i using a simple linear[2] demand function $1 - p_i^t$. Observe that base demand decreases linearly in price and recall that much of the operations/pricing literature focuses on this demand.

In addition to the base demand, we consider two terms. The first term, *the product-specific intertemporal effect*, captures the increase over base demand for product i in period t if the price is lower in this period than in period $t-1$ (which we conceptualise as a portion of the period $t-1$ market that 'waits for the period t sale'). We model this by multiplying $p_{t-1}^i - p_t^i$, the price difference between period t and the previous period, by a factor α, where α represents the degree of demand interaction across periods. This type of demand function is also considered in Ahn, Gümüş, and Kaminsky (2007) to model inter-temporal substitution.

Finally, we model the increase over base demand for product i in period t due to *substitution effects* between product i and alternative product j, and this consists of two subeffects. The first of these captures so-called *current substitution* that results if p_t^i, the price for product i in period t is less than p_t^j, the price of the alternative product. Note that the increase in demand due to inter-product substitution is captured only by the product whose price is lower in period t. Specifically, if product i is less expensive than product j in period t, then, product i is substituted for product j by consumers who initially preferred product j, but found it too expensive. We model this for both products using the positive price difference between product i and product j as a proxy for the degree of inter-product demand substitution. The second subcomponent of this term captures demand for product i in period t if its price is also lower than that of product j in the previous period, $t-1$. Similar to the first subcomponent, the increase in demand due to inter-temporal substitution is captured only by the product whose price at t is the lowest in both periods t and $t-1$. Therefore, we use the positive price difference between the lowest price in period $t-1$ and the price of product i in period t as a proxy for inter-temporal substitution effect and multiply it by the same factor α.

We combine all of these terms to develop our overall demand function. We weight the product specific and non-specific demand components by $1-\beta$ and β, respectively. Specifically, the complement of β, i.e. $1-\beta$, measures the relative weight of product specific effects and substitution effects (and thus represents product loyalty/substitutability).[3] As shown in Figure 1, this demand function has intuitively desirable characteristics: demand in period t is decreasing *piecewise linearly* in price in period t, increasing in the price difference (that is, how much lower the price is) in period t vs. the price in the previous period, and increasing in the price difference (that is, how much lower the price is) between this product and the alternative product in period t and in the previous period.

This demand function also follows from a stylised model of consumer behaviour in this system. In Appendix 1, we describe a system in which two cohorts of new customers arrive each period, half with a preference for one product, and half with a preference for the other. Each consumer has his or her on valuation for the preferred product, some fraction of consumers are loyal to their product while some are willing to substitute if their preferred product is too expensive, and some customers will stay in the market for more than one period if their valuation is not met in the first period, while others will leave after one period if their valuation is not met. In Appendix 1, we explain how this setting leads to our demand function.

Note that to facilitate subsequent analysis, intertemporal effects are limited to a single period (so that demand in period t is impacted by pricing decisions in period $t-1$, but not earlier). Also, recall that intercept and slope of the current demand component are assumed to be 1 and -1, respectively, in our demand function. However, these assumptions are not restrictive. In fact, we have performed additional analysis[4] using more complicated demand models where the intercept and slope of the current demand component are more general than 1 and -1, respectively, and product-specific and inter-temporal demand effects last for more than a single period. It turns out that most of our qualitative theoretical insights are consistent with computational testing of these more complicated demand models, although not all of them can then be proved analytically.

The retailer's objective is to find an optimal pricing plan ($\boldsymbol{p}^*$) that maximises total profit (1a) subject to constraints on the store capacity and bounds on feasible prices:

$$(\mathbf{P}_s)\ \max_{\boldsymbol{p}\geq 0} \sum_{i=1,2} \sum_{t=1}^{T} (p_t^i - c^i) D_t^i(\boldsymbol{p}) \tag{1a}$$

$$\text{s. t.} \quad \sum_{i=1,2} D_t^i(\boldsymbol{p}) \leq q \qquad \forall t \in \boldsymbol{T} \tag{1b}$$

$$0 \leq p_t^i \leq 1 \qquad \forall t \in \boldsymbol{T} \text{ and } i \in \{1, 2\} \tag{1c}$$

We use this model to develop insights into optimal promotion strategies in the presence of inter-product substitution (measured by β) and retail store capacity (measured by q). However, there are two factors that complicate the complete characterisation of optimal pricing strategies. First, the demands for both products are interrelated in each period. This inter-product dependency prevents us from decomposing the problem into two single-product problems (each of which is analyzed in Ahn, Gümüş, and Kaminsky (2007)). Secondly, the capacity constraints in each period introduce additional inter-temporal dependencies between optimal pricing decisions in different periods, which is not analytically captured in Ahn, Gümüş, and Kaminsky (2007). In order to simplify the analysis and to develop useful insights, we focus on the various limiting cases of β and q. Specifically, we first consider the uncapacitated version of our model (where $q = \infty$), and analyze two limiting cases (namely, $\beta = 0$ and $\beta = 1$) in Section 3. This enables us to better understand the effects of interproduct substitution on optimal promotion strategies and on the timing and depth of price promotions. Next, in Section 4, we consider capacitated version (where q is finite) and analyse the same two limiting cases, $\beta = 0$ and $\beta = 1$, to investigate how retail capacity affects both depth and timing of price promotions.

3. The impact of product substitution under unbounded store capacity

Recall that in addition to base demand, $D_t^i(\boldsymbol{p})$ has two additional components that model the product-specific and inter-product substitution effects, weighted by $1-\beta$ and β, respectively. In Section 3.1, we analyse a demand function exclusively consisting of product-specific effects, i.e. $\beta = 0$. Similarly, in Section 3.2, we analyse a demand model exclusively consisting of inter-product substitution effects, i.e. $\beta = 1$. This analysis gives insight into the impact of degree of substitution, β, on optimal pricing decisions – in Section 3.3 we use computational experiments to establish that the intermediate cases ($0 < \beta < 1$) fall between the two limiting cases for which we have derived closed-form solutions.

3.1 *When there is no inter-product substitution ($\boldsymbol{\beta = 0}$)*

Note that when $\beta = 0$, there is no interaction between the product, so model ($\mathbf{P}_s$) can be decomposed into two independent problems. In this benchmark case, we can employ the analysis developed by Ahn, Gümüş, and Kaminsky (2007) to characterise the optimal pricing plan as follows[5]:

PROPOSITION 1 *The retailer follows a high–low pricing strategy when there is no inter-product substitution* ($\beta = 0$) *and store capacity is unbounded* ($q = \infty$). *The optimal pricing strategy* $\forall i \in \{1, 2\}$ *involves alternating between the following high- and low prices:*

$$p^i_{hi} = \frac{1+c^i}{2} + \frac{1-c^i}{2}\left(\frac{\alpha(2+\alpha)}{4\alpha+4-\alpha^2}\right) \text{ and } p^i_{lo} = \frac{1+c^i}{2} - \frac{1-c^i}{2}\left(\frac{\alpha(2-\alpha)}{4\alpha+4-\alpha^2}\right)$$

Proposition 1 indicates that model (P_s) can be decomposed into two independent pricing problems, one for each product. Observe that in the context of this model, when there is no interaction between products, although it is feasible to use a long and complex sequence of prices, the full benefit of dynamic pricing strategy can be obtained using only two price levels for each product. Chen, Wu, and Yao (2010) reached a similar conclusion using a stochastic model.

3.2 *When there is inter-product substitution* ($\beta = 1$)

Next, we explore Model (P_s) setting $q = \infty$ and $\beta = 1$, so that there are no capacity constraints and no product-specific intertemporal effects. In contrast to the previous section, the demand functions for products i and j are interrelated. Therefore, we cannot decompose the problem into two separate subproblems. However, it is always optimal for the firm (recall that the firm is a monopolist) to price products according to their marginal costs. In other words, we can show that since $c^2 \geq c^1$, we have $p^2_t \geq p^1_t$. This implies that the firm uses pricing to segment customers, a result that is consistent with the price differentiation literature. For example, Draganska and Jain (2006) present empirical evidence that firms increase their profits by segmenting their customers based on multi-product pricing strategies. This observation allows us to express the price of product 2 in terms of the cheaper product 1, as follows:

LEMMA 1 *When* $\beta = 1$, *the optimal price of the more expensive product* p^2_t *in terms of the price of the cheaper product* p^1_t, *given* $c^2 \geq c^1$, *is*

$$p^2_t = \frac{1+p^1_t}{2} + \frac{c^2-c^1}{2}$$

By employing the above characterisation, we can transform the retailer's optimisation problem into a single-product problem defined on product 1. Once the optimal prices for product 1 are characterised, Lemma 1 can be used to obtain the optimal prices for product 2 as well. The following result provides the optimal pricing strategies for both products in closed-form:

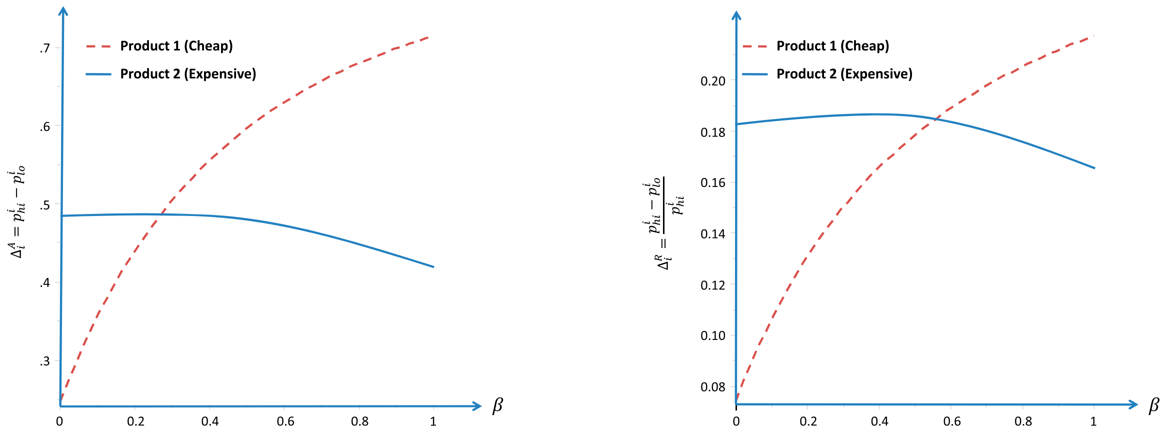
PROPOSITION 2 *The retailer follows a high–low pricing strategy when* $\beta = 1$ *and store capacity is unbounded* ($q = \infty$). *The optimal pricing strategy involves alternating between the following high and low prices:*

$$p^i_{hi} = \begin{cases} c^1 + \left[\frac{3(1-c^1)}{2} + \frac{c^2-c^1}{2}\right]\frac{2(7/2+6\alpha)}{7(7/2+4\alpha)-8\alpha^2} & \text{for } i = 1 \\ \frac{1+c^2}{2} + \left[\frac{3(1-c^1)}{2} + \frac{c^2-c^1}{2}\right]\frac{7/2+6\alpha}{7(7/2+4\alpha)-8\alpha^2} & \text{for } i = 2. \end{cases} \tag{2}$$

$$p^i_{lo} = \begin{cases} c^1 + \left[\frac{3(1-c^1)}{2} + \frac{c^2-c^1}{2}\right]\frac{2(7/2+2\alpha)}{7(7/2+4\alpha)-8\alpha^2} & \text{for } i = 1 \\ \frac{1+c^2}{2} + \left[\frac{3(1-c^1)}{2} + \frac{c^2-c^1}{2}\right]\frac{7/2+2\alpha}{7(7/2+4\alpha)-8\alpha^2} & \text{for } i = 2. \end{cases} \tag{3}$$

Observe that similar to the pattern characterised in Proposition 1, it is only necessary to use two prices for each product, and the optimal price for each product outlined in Proposition 2 consists of a base component and an adjustment term that is larger in the high-priced period than in the low priced period. However, both base price and the adjustment terms are affected by the presence of inter-product demand substitution. Specifically, while the base price component for both products under $\beta = 0$ is equal to the optimal non-interaction price[6] (i.e. $\frac{1+c^i}{2}$), the same holds true only for product 2 under $\beta = 1$. On the other hand, the base price for the product 1 is equal to its marginal cost (i.e. c^1), which is less than the optimal non-interaction price for that product. This discrepancy stems from the fact that when $\beta = 1$, product 1 is priced below its non-interaction price in order to capture the substitution demand from product 2. Also, for the same reason, the optimal depth of the markdown between two consecutive periods for product 1 is larger than that of product 2.

Given these differences between optimal prices under $\beta = 0$ and $\beta = 1$, Propositions 1 and 2 show that the effectiveness of a one-period high/one-period low pricing strategy is quite robust to the degree of interaction among products. Hence, in the next section, we restrict our attention to this specific pricing strategy, and explore how optimal high and low price levels change with respect to the degree of interaction between products, i.e. β.



(A) Absolute Difference (i.e., $\Delta_i^A = p_{hi}^i(\beta) - p_{lo}^i(\beta)$) (B) Relative Difference (i.e., $\Delta_i^R = \frac{p_{hi}^i(\beta) - p_{lo}^i(\beta)}{p_{hi}^i(\beta)}$)

Figure 2. The change in absolute and relative differences between price promotions set for the relatively cheap product (Product 1) and more expensive product (Product 2) with respect to β. In this example, we assume that $c^1 = 0.1$, $c^2 = 0.15$, and $\alpha = 1$.

3.3 *Sensitivity analysis with respect to β*

In this section, we study the effect of β, i.e. the degree of product substitution, on the depth of price promotions – the difference between high and low prices. Note that there are two alternative ways of measuring the difference: absolute difference, i.e. $\Delta_i^A = p_{hi}^i - p_{lo}^i$, and relative difference, i.e. $\Delta_i^R = \frac{p_{hi}^i - p_{lo}^i}{p_{hi}^i}$. In the next proposition, focusing on the two limiting cases of $\beta = 0$ and $\beta = 1$ characterised by Propositions 1 and 2 respectively, we compare both absolute and relative differences for the less expensive product (product 1) to those for the more expensive one (product 2):

Proposition 3 *Comparison of the depth of price promotions for the cheaper product* 1 *and the more expensive product 2:*

β	Absolute price discount	Relative price discount
$\beta = 0$ (no substitution)	$\Delta_1^A \leq \Delta_2^A$	$\Delta_1^R \leq \Delta_2^R$
$\beta = 1$ (full substitution)	$\Delta_1^A \geq \Delta_2^A$	$\Delta_1^R \geq \Delta_2^R$

Proposition 3 indicates that substitution and product-specific effects have an opposite impact on pricing strategy. If there is no interaction between the two products (labeled *no substitution*), the retailer sets a shallower discount on the cheaper product (product 1) compared to the discount set on the more expensive product 2. In contrast, if demands for two products are interrelated (labeled *full substitution*), the decision is the opposite – the retailer sets a deeper discount on the cheaper product.

The intuition behind Proposition 3 is as follows: Price promotions enable a retailer to inter-temporally differentiate high-valuation consumers from low-valuation ones for both products. However, as β increases, a greater number of both high- and low-valuation consumers switch from buying product 2 (more expensive) to product 1 (cheaper). This increases the incentive for the retailer to price discriminate for the cheaper product, and thus to increase the discount offered for this product.

Although we are only able to analytically characterise optimal prices for the two limiting cases of β, our numerical study shows that as we might expect, when $0 < \beta < 1$, the direction of discounts falls between these two extremes. For example, in Figure 2(A) and (B), we present absolute and relative differences between optimal high and low prices as β changes between 0 and 1. Figure 2 illustrates that when β is large (interaction between the two products is strong), the lower priced product has a larger price decrease than the higher priced product. However, as the degree of interaction decreases (β approaches 0), the higher priced product's price drop increases relative to that of the lower-priced product.

4. The impact of bounded store capacity

In Section 3, we discussed the impact of the degree of inter-product substitution, β, on price promotions under unbounded retail capacity. In that case, we observed that the relative timing of price promotions is unaffected by β, and hence, we focused only on the depth of promotions. However, with finite retail capacity, both timing and depth of price promotion will be affected by β, due to the relationship between capacity utilisation and pricing decisions. Specifically, capacity utilisation can enter into the problem in two ways: First, when capacity is limited, it may be optimal to reduce the depth of price promotion in order to more effectively utilise limited capacity. Second, when capacity is limited, the relative timing of promotions can crucially impact capacity utilisation.

For example, consider the case when $\beta = 0$. The alternating one-period high/one-period low pricing strategy can be implemented in two different ways: (i) the high and low prices for both products can be offered at the same time (so that one period both products have high prices, and the next period both products have low prices – we call this the *simultaneous* strategy), or (ii) the high and low prices can be offered at different times (so that in any period, one product has a high price and the other has the low price – we call this the *alternating strategy*). Even though both strategies lead to exactly the same total demand realisation for the firm, when capacity constraints are binding, they are very different in terms of capacity utilisation. The *simultaneous* strategy requires that the firm leave some of the capacity unused when the price is high (because of low demand). In contrast, the *alternating* strategy enables the firm to shift the allocation of capacity in each period between the two products, and hence, leads to more effective capacity utilisation.

Thus, regardless of demand interaction, prices for the two products are linked due to this 'capacity interaction'. Furthermore, both price levels and specific timing of relative price offerings need to be explicitly determined in conjunction with capacity utilisation decisions. A general analysis of pricing and timing of price promotions is analytically intractable, so in order to better understand the impact of capacity constraints on the timing of price changes for different products, we restrict our attention to simplified setting, infinite-horizon one-period high/one-period low pricing strategies, assuming that two products have identical marginal costs (i.e. $c^i = c$ for $i = \{1, 2\}$). As in the previous section, we consider the $\beta = 0$ and $\beta = 1$ cases, separately.

We can decompose the problem into an infinite series of the following two-period pricing problem where optimal prices for periods $t = \{1, 2\}$, i.e. $\boldsymbol{p} = (p_1^1, p_1^2, p_2^1, p_2^2)$ can be computed by solving the following mathematical program:

$$\begin{aligned} \max \sum_{i\in\{1,2\}} \sum_{t\in\{1,2\}} & D_t^i(\boldsymbol{p})(p_t^i - c) \\ \text{s. t.} \quad & \sum_{i\in\{1,2\}} D_t^i(\boldsymbol{p}) \le q, \quad \forall t \in \{1,2\} \\ & \boldsymbol{p} = (p_1^1, p_1^2, p_2^1, p_2^2) \in \{\boldsymbol{p}_s = (p_{hi}^1, p_{hi}^2, p_{lo}^1, p_{lo}^2), \boldsymbol{p}_a = (p_{hi}^1, p_{lo}^2, p_{lo}^1, p_{hi}^2)\} \\ & 0 \le p_{lo}^i \le p_{hi}^i \le 1, \quad \forall i \in \{1,2\} \end{aligned} \tag{4}$$

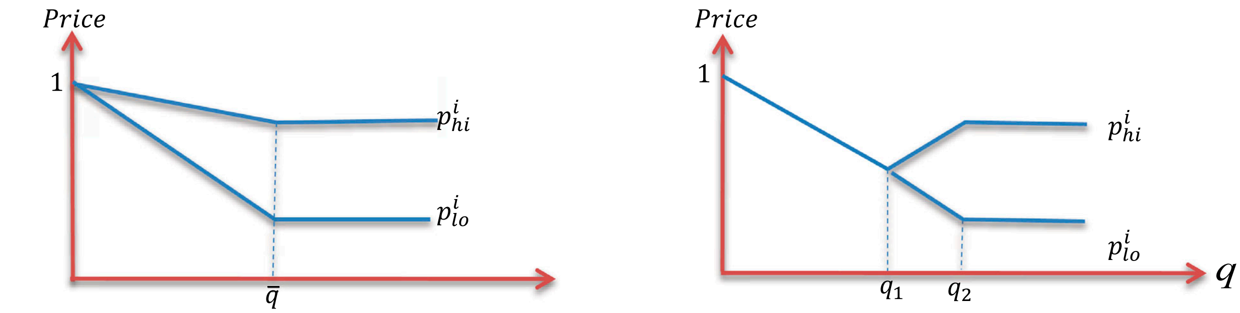
The terms in the objective function account for the total profit, the first constraint models store capacity, and the second constraint specifies either a *simultaneous* (i.e. $\boldsymbol{p}_s = (p_{hi}^1, p_{hi}^2, p_{lo}^1, p_{lo}^2)$) or an *alternating* strategy (i.e. $\boldsymbol{p}_a = (p_{hi}^1, p_{lo}^2, p_{lo}^1, p_{hi}^2)$). The final constraints specify non-negativity of the decision variables. We characterise the optimal solution to this problem as follows:

Proposition 4 *The timing of price promotions, given an infinite-horizon, stationary, one-period high and one-period low pricing model with identical marginal costs* (4)*, is as follows:*

Pricing strategy		$\beta = 0$ Alternating $\boldsymbol{p}^* = \boldsymbol{p}_a = (p_{hi}^1, p_{lo}^2, p_{lo}^1, p_{hi}^2)$	$\beta = 1$ Simultaneous $\boldsymbol{p}^* = \boldsymbol{p}_s = (p_{hi}^1, p_{hi}^2, p_{lo}^1, p_{lo}^2)$
Price of Product 1	p_{hi}^1	$1 - \frac{1}{4}\min(q, \bar{q})(2-\alpha)$	$1 - \frac{\min(q,q_1)}{2} + \frac{1}{2}[\min(q, q_2) - q_1]^+ \frac{3\alpha+4\alpha^2}{7+14\alpha+6\alpha^2}$
	p_{lo}^1	$1 - \frac{1}{4}\min(q, \bar{q})\frac{2+3\alpha-\alpha^2}{1+\alpha}$	$1 - \frac{\min(q,q_1)}{2} - \frac{1}{2}[\min(q, q_2) - q_1]^+ \frac{7+7\alpha-4\alpha^2}{7+14\alpha+6\alpha^2}$
Price of Product 2	p_{hi}^2	$1 - \frac{1}{4}\min(q, \bar{q})(2-\alpha)$	$1 - \frac{\min(q,q_1)}{4} + \frac{1}{4}[\min(q, q_2) - q_1]^+ \frac{3\alpha+4\alpha^2}{7+14\alpha+6\alpha^2}$
	p_{lo}^2	$1 - \frac{1}{4}\min(q, \bar{q})\frac{2+3\alpha-\alpha^2}{1+\alpha}$	$1 - \frac{\min(q,q_1)}{4} - \frac{1}{4}[\min(q, q_2) - q_1]^+ \frac{7+7\alpha-4\alpha^2}{7+14\alpha+6\alpha^2}$

where the closed-form expressions for $\overline{q}$, q_1, *and* q_2 *are provided in the Appendix 1.*

Proposition 4 suggests that the timing of price promotions for the products depends on the degree of substitution among them. If the degree of substitution is low ($\beta = 0$), the optimal strategy is to set the price promotions in an alternating fashion, so that each product experiences higher demand in alternating different periods, and thus the limited space can be devoted to meeting this high demand for the first product and for the second product in alternating periods. On the other hand, if



(A) Sensitivity of p^i_{hi} and p^i_{lo} w.r.t. q when $\beta = 0$ (B) Sensitivity of p^i_{hi} and p^i_{lo} w.r.t. q when $\beta = 1$

Figure 3. The sensitivity of high and low prices with respect to store capacity q when $\beta = 0$ and $\beta = 1$.

the degree of substitution is high ($\beta = 1$), this (alternating) strategy would imply that low reservation price customers for a particular product arriving during a higher price period for that product would substitute the other product, which would have a lower price in that period, thus limiting inter-temporal demand substitution. Thus, a simultaneous price promotion strategy would more successfully allow potential profits due to inter-temporal demand substitution to be captured.

Similarly, for the limiting cases $\beta = 0$ and $\beta = 1$, we can characterise the sensitivity of the depth of price promotions to changes in capacity availability.

PROPOSITION 5 *The effect of increasing store capacity* (q) *on the pricing and the depth of price promotions in both absolute and relative terms in an infinite-horizon, stationary, one-period high and one-period low pricing model* (4) *is as follows:*

Optimal Values	If degree of substitution are low ($\beta = 0$)	If degree of substitution is high ($\beta = 1$)
High Price (p^i_{hi})	decreases	first decreases and then increases
Low Price (p^i_{lo})	decreases	decreases
Absolute depth of promotion (i.e. $\Delta^A_i = p^i_{hi} - p^i_{lo}$)	increases	increases
Relative depth of promotion $\left(\text{i.e.}\, \Delta^R_i = \frac{p^i_{hi} - p^i_{lo}}{p^i_{hi}}\right)$	increases	increases

Observe in Figure 3 that depth of price promotion increases in store capacity regardless of the degree of inter-product substitution. Indeed, when capacity is tight, the retailer focuses on newly arriving demand, and so does not value the ability of deep promotions to segment customers. Also, prices are monotonically decreasing in store capacity, with the exception of the high price under $\beta = 1$. With a high degree of substitution ($\beta = 1$) and tight capacity, the high price decreases in q, whereas if there is ample capacity, the high price increases in q. This implies a preference on the part of the retailer for inter-product segmentation over inter-temporal segmentation when capacity is tight, since when $q \leq q_1$, prices in consecutive periods are kept equal to each other for each product (i.e. $p^i_{hi} = p^i_{lo}$), precluding inter-temporal segmentation. However, when there is ample capacity (i.e. $q > q_1$), the retailer increases the high price and decreases the low price, facilitating both inter-temporal and inter-product substitution.

4.1 *Summary of analytical results*

Before proceeding further, we summarise the analytical observations[7] from Sections 3 and 4. We empirically validate these observations in the next section:

- Analytical Result 1: Price promotions at high capacity stores are deeper than those at relatively low capacity stores (Proposition 5).

- Analytical Result 2: Price promotions offered on more expensive products are *deeper* than those offered on cheaper products if the degree of substitution is *low*, and *shallower* than those offered on cheaper products if the degree of substitution is *high* (Proposition 3).
- Analytical Result 3: Price promotions are scheduled *sequentially* if the degree of substitution is *low*, and scheduled *simultaneously* if the degree of substitution is *high* (Proposition 4).

5. Empirical survey

To validate our analytical observations in a more general and realistic setting, we conduct a survey of price promotions offered by four Canadian retailers (IGA, Loblaws, Metro, and Provigo within two popular CPG product categories (detergents and shampoos). We study the correlation between the depth and timing of price promotions offered by these retailers with their aisle capacity and the degree of inter-product substitution in their assortments. This survey supports the analytical results in the previous sections of this paper, subject to the limitations of our data-set, which we detail in subsequent sections.

5.1 *Data, measures and limitations*

Our data-set is based on four Canadian retailers (IGA, Loblaws, Metro, and Provigo indexed by $j \in \{1, 2, 3, 4\}$) within two popular CPG product categories (detergents and shampoos indexed by $k \in \{1, 2\}$). It includes the brand, package size and weekly price for each SKU sold by store j within category k over a period of $T = 8$ weeks in June–August 2011. The summary statistics for the retailers and product categories under consideration are provided in the Appendix 3. We construct the following measures to assess the relationship between price promotion depth and timing, capacity and substitution. (Tables D1–D3 in Appendix 4 give the summary statistics and correlations between the measures.)

5.1.1 *Depth of price promotion*

We assume that the mode of the prices over the T weeks represents the price of a product when it is *not* under promotion. If the price in a given week is less than the mode, this indicates a price promotion or sale. Let p_{ijkt} denote the price of SKU i of the product category k at store j in week t and $p_{ijkm} = \underset{t \in T}{Mode}\left(p_{ijkt}\right)$ denote the mode of the price of SKU i over $T = 8$ weeks. As in Mathur and Sinitsyn (2013), we measure the depth of a price promotion as the relative difference between the mode and the promoted price. The relative discount offered on SKU i, at store j, within category k, during week t is defined as $R_{ijkt} = \frac{(p_{ijkm} - p_{ijkt})}{p_{ijkm}}$ if $p_{ijkt} \leq p_{ijkm}$ and $R_{ijkt} = 0$ if $p_{ijkt} > p_{ijkm}$.[8] Summary statistics for R_{ijkt} are given in the Appendix 3.

5.1.2 *Capacity*

We use the aisle area devoted by a retailer to a product category as a proxy for the store capacity assigned to a product category. Let A_{jk} denote the aisle area at store j devoted to product category k. We also control for the total aisle area available in the entire store, denoted by A_j. There are obvious limitations to using aisle space for measuring capacity. This approach implicitly assumes that the shelf space allocated to a category or brand is an exogenous and static decision, whereas in practice, it is possible that this is an endogenous, dynamic decision made by retailers. Nevertheless, given the limitations of our data-set, this is our best possible measure of available capacity.

5.1.3 *Degree of substitution*

Let DS_{jk} denote the degree of substitution available at store j, within category k. We use the number of unique products carried by a store within a category as a proxy for the degree of product substitution available to consumers.[9] Once again, we acknowledge the obvious limitations of this measure of product substitution. Depending on the product category, the number of unique products may be more reflective of the degree of segmentation of consumer needs within that category. However, it is reasonable to assume that the degree of product substitution available to consumers is likely to increase as the number of unique products stocked by a store increases, and given the limitations of our data-set, this appears to be the best proxy available.

Table 1. Variables, descriptions and measurements.

Variable name	Description/measurement
i, j, k	Indices denoting a SKU, a retailer, a product category respectively
t	Index denoting a time-period
T	Total number of time-periods
p_{ijkt}	Price of SKU i sold at retailer j within category k during time-period t
p_{ijkm}	Mode of Price of SKU i sold at retailer j within category k during all time-periods $t \in T$
R_{ijkt}	Relative discount offered on SKU i sold at retailer j within category k during time-period t
A_{jk}	Aisle area devoted by retailer j towards stocking products of category k
A_j	Total aisle area available at retailer j
DS_{jk}	Degree of Substitution available in the retail assortment at store j, within category k, measured as the number of unique products carried
C_k	Indicator variable representing category k
R_j	Indicator variable representing retailer j
M_{jk}	# SKU carried by store j, within category k
u_{ijkt}	Normalised price of SKU i sold at retailer j within category k during time-period t, measured as the ratio of price and package size
U_{ijkt}	Indicator variable denoting if SKU i is categorised as expensive or inexpensive
Y_t, Z_t	Indicator variables denoting whether SKU y, z are, respectively, under promotion during week t
$F_t(Y_t, Z_t)$	Function measuring whether SKU y, z are simultaneously promoted or not during week t
Δ_{yjk}	Extent to which SKU y at store j in category k is simultaneously promoted relative to all the other SKUs within its category

5.1.4 *Normalised prices*

We also need a measure of how expensive a product is relative to all products sold within its product category. We do this by comparing the normalised price of a product to the median normalised price of all products sold within its category. Let u_{ijkt} represent the normalised price of SKU i sold at store j in category k at time t, measured as the ratio of its price and package size. We define an indicator variable U_{ijkt} where $U_{ijkt} = 1$, if $u_{ijkt} > \underset{i}{Median}\left[u_{ijkt}\right]$, $U_{ijkt} = 0$ otherwise. Thus, $U_{ijkt} = 1$ categorises a product as relatively expensive, while $U_{ijkt} = 0$ categorises it as relatively cheap. Note that a binary classification of products is consistent with the theoretical model of two representative products that are analysed in the previous section.

5.1.5 *Simultaneous price promotions*

We measure the extent to which price promotions within a product category at a store occur simultaneously. Define Y_t and Z_t to be indicator variables denoting whether SKUs y and z are under promotion during week t. We develop a weighting function $F_t(Y_t, Z_t)$ that measures the extent to which SKUs y and z are simultaneously promoted over time $t \in T$. Given a pair of products, four distinct states are possible: If neither SKU is promoted at time t, then $F_t(0, 0) = 0$; if the SKUs are simultaneously promoted at time t, then $F_t(1, 1) = 1$; and if only one of the products is promoted at time t, then either $F_t(0, 1) = -\frac{1}{2}$ or $F_t(1, 0) = -\frac{1}{2}$. Setting the weights in this way implies that $E[F_t(Y_t, Z_t)] = 0$ if the four states of price promotion are equally likely. We define the function $F_t(Y_t, Z_t) = 2Y_tZ_t - \frac{Y_t+Z_t}{2}$ to generate these weights. The sum $S_{yz} = \sum_{t=1}^{T} F_t(Y_t, Z_t)$ characteriSes the extent to which SKUs y and z are simultaneously promoted over the T period horizon, where $\left(-\frac{T}{2} \leq S_{yz} < T\right)$, and for ease of exposition, we assume that $S_{yy} \equiv 0$. We let Δ_{yjk} be a measure of the average degree of simultaneous promotion between SKU y at store j in category k each of the other SKUs sold within the same category at the same store, and let M_{jk} denote the total number of SKUs at store j in category k. Then, Δ_{yjk} is defined as follows.

$$\Delta_{yjk} = \frac{\sum_{z=1}^{M_{jk}} S_{yz}}{M_{jk} - 1} \tag{5}$$

Table 1 summarises the notation used in the empirical study.

5.1.6 *Additional limitations*

Our empirical survey assumes that the primary incentive for retailers to offer price promotions is to improve their ability to segment high- and low-valuation consumers. However, there are other motives driving price promotions. For example, price promotions may be utilised to manage excess inventory, induce new-product trial and manage brand image. Our subsequent regression analysis does not control for these alternate explanations due to the limitations of our data-set.

5.2 *Model and estimation*

We specify two regression models, focusing first on the depth and second on the timing of price promotions. First, we model the impact of retail capacity and extent of product substitution in retail assortment on the *depth* of price promotions:

$$R_{ijkt} = \beta_0 + \beta_1 A_{jk} + \beta_2 A_j + \beta_3 U_{ijkt} + \beta_4 DS_{jk} + \beta_5 U_{ijkt} DS_{jk} + \beta_6 C_k + \gamma_{i,t} + \varepsilon_{i,t} \tag{6}$$

Observe that in addition to terms for capacity, degree of substitution and relative expense of each product, we control for differences in price promotions across product categories, as well as for the differences across retailers and the brand for each SKU. To control for differences in promotions across categories, we define C_k to be a dummy variable for the product category, where $C_k = 1$ denotes a detergent, while $C_k = 0$ denotes a shampoo. To control for differences across retailers, we include the total aisle space available at store j, A_j, as a regressor.

Next, recall that Δ_{ijk} is a measure of the extent to which SKU i at store j in category k is simultaneously promoted with respect to the other SKUs sold within the same category and store as SKU i. We specify the following model to investigate how the extent of product substitution in retail assortment influences the *timing* of price promotions.

$$\Delta_{ijk} = \beta_7 + \beta_8 DS_{jk} + \beta_9 C_k + \sum_{j'=1}^{3} \beta_{10,j'} R_{j'} + \gamma_{i,t} + \varepsilon_{i,t} \tag{7}$$

DS_{jk} and C_k are as defined in the previous model. R_j is a dummy variable indicating that SKU i was sold at retailer $j \in \{1, 2, 3\}$.[10]

We use mixed effect models to analyse our data. A key feature of mixed models is that both random effects and fixed effects can be specified to extract multiple sources of variation. For instance, a given brand is often sold in different sizes and flavours. The price promotions offered on products belonging to the same parent brand (e.g. Pantene) but different flavours (e.g. Pantene Pro-V Classic Care, Pantene Pro-V Flat to Volume) may be more correlated than the price promotions offered across competing brands (e.g. Pantene vs. Sunsilk). Also, it is possible that trade promotions offered by manufacturers may influence the retail prices and therefore the resulting price promotions, and there may be other sources of variation. In an effort to control for such common variation within brands, we estimated the brand coefficients $\gamma_{i,t}$ in both models shown above as a random effect, where $\gamma_{i,t} = \lambda_i Brand_{i,t}$ and $\lambda \sim N(0, \sigma_i^2)$ are a vector of random parameters.

We estimate the mixed effects models as follows. The models have the following general form: $Y = X\beta + Z\lambda + \varepsilon$, where $\lambda \sim N(0, \Psi)$ and $\varepsilon \sim N(0, \Lambda)$. Here, X corresponds to the fixed effects, β to the fixed-effect coefficients, Z to the random effect (brand) and λ to the random-effect coefficients. Ψ is the covariance matrix of the random effects and Λ is the covariance matrix of the error term. We estimate the mixed effect regression models using the restricted maximum likelihood (REML) estimation method. REML estimators are obtained by maximising only the part of the likelihood function that is invariant to the fixed effects in the linear model rather than the complete likelihood function. As a result, REML estimators take into account the loss of degrees of freedom in estimating the mean and therefore, produce unbiased estimators for the variance parameters (Smyth and Verbyla 1996).

5.3 *Results*

We estimate three nested models – a model with intercept only, a model with fixed effects only and a model with random effects only. Finally, we estimate the full model incorporating both fixed- and random effects. The regression coefficients are presented in Tables 2 and 3. The full model incorporating both fixed- and random effects yields the best fit with the data, based on measures of log-likelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC), demonstrating the value of estimating a mixed-model.

Observe in Table 2 that $\beta_1 = 0.004$. Based on Proposition 5, price promotions at high capacity stores are deeper than those at relatively low capacity stores, we hypothesise that β_1 should be greater than 0. We observe that β_1 is positive, as hypothesised, and moreover, it is statistically significant ($p < 0.001$). Thus, we find empirical evidence in support of Proposition 5. Similarly, our empirical prediction based on Proposition 3 is that price promotions offered on expensive products are deeper than those offered on cheaper products if the degree of substitution is low and shallower than those offered on cheaper products if the degree of substitution is high. Based on Proposition 3, we hypothesise that $\beta_5 < 0$, since a negative coefficient implies negative interaction between the degree of substitution and whether a product is cheap or expensive. As can be seen in Table 2 regression analysis estimates $\beta_5 = -0.002$, and furthermore it is statistically significant

Table 2. Coefficient estimates for model (6).

	Fixed effects							
	(a)[a]		(b)		(c)		(d)	
	β	SE	β	SE	β	SE	β	SE
$Intercept(\beta_0)$	0.186***	0.004	0.169***	0.032	0.193***	0.016	0.139***	0.038
$A_{jk}(\beta_1)$			**0.004****	**0.001**			**0.004****	**0.001**
$A_j(\beta_2)$			−0.0003***	0.000			−0.0003***	0.000
$U_{ijkt}(\beta_3)$			0.147***	0.024			0.195***	0.026
$DS_{jk}(\beta_4)$			0.003***	0.000			0.004***	0.000
$U_{ijkt}DS_{jk}(\beta_5)$			**−0.002*****	**0.000**			**−0.002*****	**0.000**
$C_k(\beta_6)$			0.037	0.015			0.020	0.032
	Random Effects							
	γ	SE	γ	SE	γ	SE	γ	SE
Brand					0.007**	0.002	0.008**	0.003
−2 LL	−990		−1063.6		−1131.3		−1222.5	
AIC	−988		−1061.6		−1127.3		−1218.4	
BIC	−983.5		−1057.1		−1124.3		−1215.5	

[a]Description of nested models: (a) Intercept only, (b) Fixed effects only, (c) Random effects only, (d) Full Model: Fixed- and random effects.
$^*p \leq 0.05$; $^{**}p \leq 0.01$; $^{***}p \leq 0.001$.

Table 3. Coefficient estimates for model (7).

	Fixed effects							
	(a)[a]		(b)		(c)		(d)	
	β	SE	β	SE	β	SE	β	SE
$Intercept(\beta_7)$	0.557***	0.013	0.498***	0.063	0.560***	0.033	0.448***	0.067
$DS_{jk}(\beta_8)$			**0.003*****	**0.001**			**0.003*****	**0.001**
$C_k(\beta_9)$			−0.243***	0.029			−0.170***	0.049
$R_1(\beta_{10,1})$ (IGA)			0.041	0.035			0.072*	0.034
$R_2(\beta_{10,2})$ (Loblaws)			−0.093*	0.042			−0.098*	0.040
$R_3(\beta_{10,3})$ (Metro)			−0.112**	0.038			−0.105**	0.037
	Random effects							
	γ	SE	γ	SE	γ	SE	γ	SE
Brand					0.032***	0.009	0.015**	0.005
-2 LL	427.9		258.1		236.0		201.9	
AIC	429.9		260.1		240.0		205.9	
BIC	434.4		264.5		243.5		209.4	

[a]Description of nested models: (a) Intercept only, (b) Fixed effects only, (c) Random effects only, (d) Full Model: Fixed- and random effects.
$^*p \leq 0.05$; $^{**}p \leq 0.01$; $^{***}p \leq 0.001$.

($p < 0.001$), supporting Proposition 3. Finally, note that Proposition 4 implies that price promotions should be scheduled sequentially if the degree of substitution is low, and scheduled simultaneously if the degree of substitution is high and so if $\beta_8 > 0$, the field evidence will be consistent with Proposition 4. In Table 3, we see that $\beta_8 = 0.003$, and thus is positive as hypothesised, and moreover, it is statistically significant ($p < 0.001$). Thus, we also find empirical evidence in support of Proposition 4.

Our results are summarised below:

(1)	Price promotions at high capacity stores are deeper than those at relatively low capacity stores .
(2)	Price Promotions offered on more expensive products are *deeper* than cheaper products if the degree of substitution is *low*.
	Price Promotions offered on more expensive products are *shallower* than cheaper products if the degree of substitution is *high*.
(3)	Price promotions are scheduled *sequentially* if the degree of substitution is *low*.
	Price promotions are scheduled *simultaneously* if the degree of substitution is *high*.

Thus, the theoretical model analysis summarised in Section 4.1 yields promotion strategies that are consistent with the three empirical patterns outlined above.

6. Conclusion

In this paper, we explore, both empirically and analytically, how the interplay of inter-temporal and inter-product substitution, price sensitivity and store capacities impacts the timing and depth of sales promotions. Our analysis suggests that the value a firm can generate by exploiting inter-temporal and substitution effects is impacted by the amount of excess capacity it possesses, as well as the degree of substitution. Specifically, when there is a great deal of possible substitution between products, a firm is typically better off offering simultaneous sales (that is, low prices) for multiple products and offering larger price reductions for the less expensive products, whereas when there is little substitution, alternating sales and large price reductions for the more expensive products leads to higher profit. Also, the depth of price promotion depth should increase in available retail capacity.

In general, we see that promotion decisions generate significant benefit if they are aligned with the degree of substitution across products and time as well as with retail capacity. However, we did not explicitly consider “menu costs” associated with changing prices and disseminating promotion information, and these will ultimately play a role in the design and implementation of effective promotion strategies. Of course, even if potential gains from inter-temporal price differentiation dominate menu costs, retailers must design promotion strategies that minimise the cannibalisation of demand for other products, and that account for capacity limitations.

In addition to omitting menu costs, our analysis has a variety of limitations. Our analytical model is clearly highly stylised. For example, we consider only two products, ignore competitive effects, and assume that demand functions are deterministic. Our empirical analysis similarly has limitations. Our analysis is based on data from only four firms and two product categories, and necessarily uses a potentially imprecise measures for degree of substitution and available capacity. Nevertheless, we believe that our analysis produces useful guidelines for promotion timing and level, and solid insights into why these guidelines make sense.

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Notes

1. Corsten and Gruen (2003), in general, find a 2:1 ratio between promoted and non-promoted out-of-stock rates.

2. Linear demand functions have been commonly used in the operations management literature – see Lus and Muriel (2009) and references therein. Although these linear models are not based directly on individual consumer choice models, their aggregate linear form has both theoretical and empirical support. See Singh and Vives (1984) for an analytical derivation of linear demand functions from the maximisation problem of a representative consumer with a quadratic concave utility function. Also, using the data collected from supermarket chains, (Pesendorfer 2002) empirically validates the price-elasticity structure of such a linear demand function.
3. This parameterisation of product substitutability via β has been also commonly used in both economics and operations management literatures – see Röller and Tombak (1990, 1993), Goyal and Netessine (2007) and references therein.
4. This additional analysis is available upon request from the authors.
5. All the proofs are provided in the Appendix 2.
6. Note that $\frac{1+c^i}{2}$ is the optimal price for product i if the monopolist ignores all interactions between products and time periods, i.e. both α and β are equal to zero.
7. In addition to these analytical results, we have conducted an extensive computational study to explore the value of explicitly considering product-specific intertemporal effects and substitution effects when making operational decisions. We refer reader to Appendix B.3 for a detailed discussion of the design and implementation of computational study, as well as the results of this study.
8. Alternately, the depth of a price promotion can be measured with respect to the maximum price $\text{Max}_{t\in T}\left(p_{ijkt}\right)$, as $R^{'}_{ijkt} = \frac{\text{Max}_{t\in T}\left(p_{ijkt}\right)-p_{ijkt}}{\text{Max}_{t\in T}\left(p_{ijkt}\right)}$. We repeated our analysis using this alternate measure and found the same qualitative results.
9. An alternate measure for the degree of substitution available within a product category is the number of SKUs carried by store j within category k, denoted by M_{jk}. We repeated the empirical analysis using this alternate measure and find the same qualitative results. The number of unique products DS_{jk} is smaller than M_{jk} since the number of unique products was measured by counting the SKUs of different package sizes of the same brand as a single unique product.
10. The retailer Provigo ($j = 4$) is the reference group.
11. Proofs of Lemma 7 and 8 are available from the authors upon request.

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Appendix 1. Consumer behaviour model

In this section, we briefly discuss a stylised consumer behaviour model that leads to the demand function $D_t^i(\boldsymbol{p})$ employed in this paper. We assume that a total of $2N$ new consumers arrive each period, where without loss of generality, we can normalise $N = 1$. The consumers are heterogeneous along four dimensions. First, consumers are horizontally differentiated, as follows. In each period, half of the new consumers prefer product 1 to product 2, while the remaining half of the new consumers prefer product 2 to product 1. Second, consumers are heterogenous in their loyalty levels, as follows. They are divided into *loyals* and *switchers*. A *loyal* prefers a particular product (either product 1 or 2) and never considers the substitute product. In contrast, a *switcher* also prefers a particular product (either product 1 or 2), but may switch and purchase the substitute product if the preferred product is too expensive. A fraction β of consumers are *loyals*, and the remaining fraction $1 - \beta$ are *switchers*. Thus, there are four segments of new consumers who arrive in each period: (i) β product 1 *loyals*; (ii) $(1 - \beta)$ product 1 switchers (that is, they prefer product 1 but may switch to product 2; (iii) β product 2 *loyals*; (iv) $(1 - \beta)$ product 2 *switchers*.

Third, consumers are vertically differentiated. Each consumer has a valuation v that is uniformly distributed between 0 and 1. Fourth, consumers are heterogenous in the maximum time they stay in the market. An α fraction of consumers stay in the market for up to two periods, while the remaining $1 - \alpha$ fraction of consumers stay in the market for a single time period. If their valuation is not met during this time, they leave the market.

A newly arrived *switcher* makes a purchase decision as follows: (i) She buys her preferred product in the current period if her valuation exceeds product price; (ii) otherwise, she switches and purchases the substitute product in the current period if her valuation exceeds the price of that product; (iii) otherwise, with probability α she waits for the next period and buys the less expensive product provided that her valuation exceeds the price, and with probability $1-\alpha$ leaves the market; (iv) otherwise, she leaves the market.

We refer to *switchers* who wait for the next period as *residual switchers*. If residual switchers remain in the market for two periods and purchase in the second period, they purchase the less expensive product. If the prices happen to be the same in the next period, they buy either product with equal probability. Let $K_t^i(\boldsymbol{p})$ be the demand generated by switchers for product i during period t, expressed as follows.

$$K_t^i(\boldsymbol{p}) = \begin{cases} \beta(1-p_t^i) + \beta\left[p_t^j - p_t^i\right]^+ + \beta\alpha\left[\min(p_{t-1}^i, p_{t-1}^j) - p_t^i\right]^+ & \text{if } p_t^i \le p_t^j \\ \beta(1-p_t^i) & \text{if } p_t^i > p_t^j \end{cases} \tag{A1}$$

where $i, j \in \{1, 2\}$, and $i \neq j$.

In contrast to *switchers*, *loyals* never purchase the substitute product. A newly arrived *loyal* makes a purchase decision as follows: (i) She buys her preferred product in the current period if her valuation exceeds the product price; (ii) otherwise, she either waits with probability α for the next period and buys the product if her valuation exceeds the next periods price, or leaves the market with probability $1-\alpha$; (iii) finally, if she has been in the market for two periods and has not observed a price that is below her valuation, she leaves the market. We refer to *loyals* who wait for the next period as *residual loyals*. Let $L_t^i(\boldsymbol{p})$ be the demand generated by *loyals* for product i during period t, expressed as follows.

$$L_t^i(\boldsymbol{p}) = (1-\beta)(1-p_t^i) + (1-\beta)\alpha\left[p_{t-1}^i - p_t^i\right]^+ \tag{A2}$$

The total demand at period t is therefore equal to $D_t^i(\boldsymbol{p}) = K_t^i(\boldsymbol{p}) + L_t^i(\boldsymbol{p})$, which can be rewritten as follows:

$$D_t^i(\boldsymbol{p}) = \overbrace{1-p_t^i}^{\text{Current Demand}} + (1-\beta)\overbrace{\alpha\left[p_{t-1}^i - p_t^i\right]^+}^{\substack{\text{Product Specific}\\\text{Intertemporal Effect}}} + \beta\overbrace{\left[\overbrace{\left[p_t^j - p_t^i\right]^+}^{\text{Current Substitution}} + \alpha\overbrace{\left[\min(p_{t-1}^i, p_{t-1}^j) - p_t^i\right]^+}^{\text{Intertemporal Substitution}}\right]}^{\text{Substitution Effects}}$$

where $[x]^+ = \max(x, 0)$, $i, j \in \{1, 2\}$ and $i \neq j$.

Appendix 2. Proofs of Propositions and Lemmas

B.1 *Proof of Proposition 1*

Recall that when $\beta = 0$, there is no interaction between the products. Hence, we can decompose the problem into two independent pricing problems, one for each product $i = 1, 2$ as follows:

$$\max_{\boldsymbol{p}\ge 0}\left[(p_1^i - c^i)(1-p_1^i) + \sum_{t=2}^{T}(p_t^i - c^i)\left((1-p_t^i) + \alpha\left[p_{t-1}^i - p_t^i\right]^+\right)\right] \tag{B1a}$$

The above problem is analysed in Ahn, Gümüş, and Kaminsky (2007). We refer the readers to Theorem 3 in Ahn, Gümüş, and Kaminsky (2007) for the characterisation of the optimal pricing decision for the above problem. □

B.2 *Proof of Lemma 1*

There are two parts in this Lemma. First, we show that optimal price for product 2 is greater than that for product 1 for all time periods. Suppose that it is not true, i.e. there exists a time period t, where $p_t^1 \ge p_t^2$. Note that replacing price of product 2 with the price of product 1, i.e. $p_t^1 \leftarrow p_t^2$ and $p_t^2 \leftarrow p_t^1$ will not change the total residual and substitute demands. Therefore, the exchange will always increase the total profit since the change in total demand and margin is positive. This implies that the retailer's profit function $\pi_R(\boldsymbol{p})$ can be expressed as follows:

$$\pi_R(\boldsymbol{p}) = \left[\sum_{t=1}^{T}(p_t^2 - c^2)(1-p_t^2) + \sum_{t=1}^{T}(p_t^1 - c^1)\left((1-p_t^1) + (p_t^2 - p_t^1)\right) + \sum_{t=2}^{T}2(p_t^1 - c^1)\alpha\left[p_{t-1}^1 - p_t^1\right]^+\right]$$

Given product 1's price, we can express the optimal price of product 2 by equating the derivative of retailer's profit function with respect to p_t^2 to zero and solving it for p_t^2:

$$\frac{\partial \pi_R(\boldsymbol{p})}{\partial p_t^2} = 0 \rightarrow p_t^2 = \frac{1+p_t^1}{2} + \frac{c^2 - c^1}{2}$$

Next, we show that it is indeed never optimal for the retailer to charge same price for both products, i.e. $p_t^1 \neq p_t^2$ for all t. Suppose that it is not correct, i.e. there exists a pricing plan that has both products that have same price at time period t. We will show that there exists another pricing plan with higher profit, where the product prices are strictly different from each other. Without loss of generality, let's pick product 2. Now, we increase price of product 2 in period t by ϵ. Note that increasing it by ϵ would not change the total demand realisation at period t since demand lost by product 1 will be recovered by the product 1. On the other hand, this strictly increases the total profit at period t since some non-zero fraction of the demand paid p_t^2 previously and now pays $p_t^2 + \epsilon$. In order to complete the proof, we need to find a feasible ordering policy and show that this does not increase the cost. In order to do this, we just take the capacities that were originally used to satisfy demand for the product in 2 and shift them to product 1. Note that the assumption $c_t^1 \leq c_t^2$ ensures that shifting capacity to product 1 always leads to a plan with lower cost. □

B.3 *Proof of Proposition 2*

Recall that using Lemma 1, we can transform the problem into single-product problem. However, this single-product pricing problem turns out to be quite complicated since the form of the objective function depends both on the level and the relative order of prices of product 1. Therefore, we impose an additional constraint, denoted by π_t, that would impose a particular order on the prices of product 1 over time.

$$p^1_{\pi_1} \leq \ldots \leq p^1_{\pi_t} \leq \ldots \leq p^1_{\pi_T} \tag{B2}$$

where $\pi_t = \hat{t}$ if the minimum price at period $\hat{t}$ is the t^{th} lowest price among all T product 1 prices. Imposing the constraints in (B2), next, we show that retailer's profit function becomes concave in prices.

LEMMA 2 *When $\beta = 1$, model ($\boldsymbol{P}_s$), together with the addition of linear constraints implied by π_t (i.e. constraints provided in (B2)), is a concave optimisation problem.*

Proof of Lemma 2 For notational simplicity, we define, $p_{(t,k,i)} = \begin{cases} p_t^2, & \text{if } k = 0; \\ p_{t-1}^1, & \text{if } k > 0. \end{cases}$ and

$$I_{(t,k,i)} = \begin{cases} 1, & \text{if } k = 0 \text{ and } i = 1; \\ 1, & \text{if } k > 0;\ i = 1 \text{ and } p_t^1 \leq p_{t-1}^1; \\ 0, & \text{o.w.} \end{cases}$$

Also let $\underline{(t,k,i)}$ and $\overline{(t,k,i)}$ be the time and product indices of $p_{(t,k,i)}$. Finally, we let $\alpha^0 = 1$ and combine the last two terms in $D_t^i(\boldsymbol{p})$ by taking the summation on k from 0 to $\min(1, t-1)$ as follows:

$$D_t^i(\boldsymbol{p}) = (1 - p_t^i) + \sum_{k=0}^{\min(1,t-1)} \alpha^k (p_{(t,k,i)} - p_t^i) I_{(t,k,i)}$$

Let $f(\boldsymbol{p})$ be the retailer's revenue given a pricing plan $\boldsymbol{p}$, i.e.

$$f(\boldsymbol{p}) = \sum_{i=1}^{2}\sum_{t=1}^{T} D_t^i(\boldsymbol{p}) p_t^i$$

Let F_π be the set of prices that satisfy fixed ordering constraints π_t, i.e.

$$F_\pi = \{\boldsymbol{p} | p^1_{\pi_1} \leq \ldots \leq p^1_{\pi_t} \leq \ldots \leq p^1_{\pi_T}\}$$

In order to show that $f(\boldsymbol{p})$ is concave in $p \in F_\pi$, it suffices to prove that for any fixed $\bar{\boldsymbol{p}} \in F_\pi$, the tangent line at $f(\bar{\boldsymbol{p}})$ always lies above $f(\boldsymbol{p})$ for all $\boldsymbol{p}$ in F_π. In the next Lemma, we prove this. Therefore, $f(\boldsymbol{p})$ is concave over F_π, so that for any particular set of fixed ordering constraints, the corresponding problem to determine an optimal price plan becomes a concave maximisation problem with the set of linear constraints.

LEMMA 3 *Under any set of fixed ordering constraints π_t, $f(\boldsymbol{p}) - f(\bar{\boldsymbol{p}}) \leq \nabla f(\bar{\boldsymbol{p}})(\boldsymbol{p} - \bar{\boldsymbol{p}})$ for all $\boldsymbol{p} \in F_\pi$.*

Proof of Lemma 3 Note that $f(\boldsymbol{p}) - f(\bar{\boldsymbol{p}})$ can be expressed as follows:

$$\begin{aligned} f(\boldsymbol{p}) - f(\bar{\boldsymbol{p}}) = & \sum_{i=1}^{2}\sum_{t=1}^{T}\left(\left[p_t^i - \bar{p}_t^i\right] - \left[(p_t^i)^2 - (\bar{p}_t^i)^2\right]\right) + \sum_{i=1}^{2}\sum_{t=1}^{T}\sum_{k=0}^{min(t-1,1)} \alpha^k \left[p_t^1 p_{(t,k,i)} - \bar{p}_t^1 \bar{p}_{(t,k,i)}\right] I_{(t,k,i)} \\ & - \sum_{i=1}^{2}\sum_{t=1}^{T}\sum_{k=0}^{min(t-1,1)} \alpha^k \left[(p_t^1)^2 - (\bar{p}_t^1)^2\right] I_{(t,k,i)}. \end{aligned} \tag{B3}$$

Noting that each residual revenue term, $\alpha^k p_t^1 p_{(t,k,i)} I_{(t,k,i)}$, in $f(\boldsymbol{p})$ will contribute to $\frac{\partial f}{\partial p_t^1} = \ldots + \alpha^k p_{(t,k,i)} I_{(t,k,i)} + \ldots$ and $\frac{\partial f}{\partial p_{(t,k,i)}} = \ldots + \alpha^k p_t^1 I_{(t,k,i)} + \ldots$. From this, we know that each term, $\alpha^k p_t^1 p_{(t,k,i)} I_{(t,k,i)}$, contributes two terms to $\nabla f(\bar{\boldsymbol{p}})(\boldsymbol{p} - \bar{\boldsymbol{p}})$ through $\alpha^k \bar{p}_{(t,k,i)} \left[p_t^1 - \bar{p}_t^1\right] I_{(t,k,i)}$ and $\alpha^k \bar{p}_t^1 \left[p_{(t,k,i)} - \bar{p}_{(t,k,i)}\right] I_{(t,k,i)}$.

After some algebraic manipulation, we have

$$\nabla f(\bar{\boldsymbol{p}})(\boldsymbol{p}-\bar{\boldsymbol{p}}) = \sum_{i=1}^{2}\sum_{t=1}^{T}(\left[p_t^i - \bar{p}_t^i\right] - \left[2\bar{p}_t^i p_t^i - 2(\bar{p}_t^i)^2\right]) - \sum_{i=1}^{2}\sum_{t=1}^{T}\sum_{k=0}^{min(t-1,1)} \alpha^k \left[2\bar{p}_t^1 p_t^1 - 2(\bar{p}_t^1)^2\right] I_{(t,k,i)}$$
$$+ \sum_{i=1}^{2}\sum_{t=1}^{T}\sum_{k=0}^{min(t-1,1)} \alpha^k \left[\bar{p}_t^1 p_{(t,k,i)} + p_t^1 \bar{p}_{(t,k,i)} - 2\bar{p}_{(t,k,i)}\bar{p}_t^1)\right] I_{(t,k,i)} \tag{B4}$$

When we subtract (B4) from (B3), cancel the appropriate terms and complete the squares, we obtain:

$$f(\boldsymbol{p}) - f(\bar{\boldsymbol{p}}) - \nabla f(\bar{\boldsymbol{p}})(\boldsymbol{p}-\bar{\boldsymbol{p}}) = -\sum_{i=1}^{2}\sum_{t=1}^{T}\left[p_t^i - \bar{p}_t^i\right]^2 - \sum_{i=1}^{2}\sum_{t=1}^{T}\sum_{k=0}^{min(t-1,1)} \alpha^k \left[p_t^1 - \bar{p}_t^1\right]^2 I_{(t,k,i)}$$
$$+ \sum_{i=1}^{2}\sum_{t=1}^{T}\sum_{k=0}^{min(t-1,1)} \alpha^k \left[(p_t^1 - \bar{p}_t^1)(p_{(t,k,i)} - \bar{p}_{(t,k,i)})\right] I_{(t,k,i)}$$

We split the first and second terms into two halves as follows:

$$f(\boldsymbol{p}) - f(\bar{\boldsymbol{p}}) - \nabla f(\bar{\boldsymbol{p}})(\boldsymbol{p}-\bar{\boldsymbol{p}}) = -\frac{1}{2}\sum_{i=1}^{2}\sum_{t=1}^{T}\left[\left[p_t^i - \bar{p}_t^i\right]^2 - \left[p_t^i - \bar{p}_t^i\right]^2 - \sum_{k=0}^{min(t-1,1)} \alpha^k \left[p_t^1 - \bar{p}_t^1\right]^2 I_{(t,k,i)}\right]$$
$$-\frac{1}{2}\sum_{i=1}^{2}\sum_{t=1}^{T}\sum_{k=0}^{\min(t-1,1)} \alpha^k \left[p_t^1 - \bar{p}_t^1\right]^2 I_{(t,k,i)} + \sum_{i=1}^{2}\sum_{t=1}^{T}\sum_{k=0}^{min(t-1,1)} \alpha^k \left[(p_t^1 - \bar{p}_t^1)(p_{(t,k,i)} - \bar{p}_{(t,k,i)})\right] I_{(t,k,i)}$$

We complete our proof by showing that each strictly positive term in the last summation is outweighed by two corresponding negative terms from the first three lines.

First consider terms with $k = 0$ in the last line, i.e. $\alpha^0\left[(p_t^1 - \bar{p}_t^1)(p_{(t,0,i)} - \bar{p}_{(t,0,i)})\right] I_{(t,0,i)}$. Note that $\overline{(t,k,i)} = i$ when $k = 0$. Therefore, selecting $-\frac{1}{2}\alpha^0\left[p_t^1 - \bar{p}_t^1\right]$ and $-\frac{1}{2}\alpha^0\left[(p_t^i - \bar{p}_t^i)\right]$ from the first two lines, completing the squares, we obtain: $-\frac{1}{2}\alpha^0\left[p_t^1 - \bar{p}_t^1 - p_t^i + \bar{p}_t^i\right]^2 I_{(t,0,i)}$. Now, consider the terms with $k = 1$ in the last line. We pair $\alpha[(p_t^1 - \bar{p}_t^1)(p_{(t,k,i)} - \bar{p}_{(t,k,i)})]I_{(t,k,i)}$ from the last line with the corresponding $-\frac{1}{2}\alpha\left[p_t^1 - \bar{p}_t^1\right]^2 I_{(t,k,i)}$ from the second line. Since $\underline{(t,k,i)} = t-1$, the following term, $-\frac{1}{2}\alpha\left[p_{(t,k,i)} - \bar{p}_{(t,k,i)}\right]^2 I_{(t,k,i)}$ can be selected from the second line. Completing the squares, we obtain: $-\frac{1}{2}\alpha^0 \left[p_t^1 - \bar{p}_t^1 - (p_{(t,k,i)} - \bar{p}_{(t,k,i)})\right]^2 I_{(t,k,i)}$.

Since every positive cross product term can be matched with two corresponding negative terms, a little algebra shows:

$$f(\boldsymbol{p}) - f(\bar{\boldsymbol{p}}) - \nabla f(\bar{\boldsymbol{p}})(\boldsymbol{p}-\bar{\boldsymbol{p}}) \leq -\frac{1}{2}\sum_{i=1}^{2}\sum_{t=1}^{T}\sum_{k=0}^{min(t-1,1)} \alpha^k \left[p_t^1 - \bar{p}_t^1 - p_{(t,k,i)} + \bar{p}_{(t,k,i)}\right]^2 I_{(t,k,i)} \leq 0$$

□

Using the above result, we can generalise the 'price cycle' concept introduced in Ahn, Gümüş, and Kaminsky (2007) to the case when there is interaction between the demand for different products, $\beta = 1$. More specifically, we consider the sequence made up of the minimum price over all products in each period, and define a **minimum price cycle** as a decreasing subsequence of this 'minimum price sequence', which leads to a well defined unit of analysis for characterising the optimal dynamic pricing strategy when $\beta = 1$. Indeed, for a price sequence of length n, we can fully characterise optimal prices as follows:

LEMMA 4 *Consider model ($\boldsymbol{P}_s$) with $q = \infty$ and $\beta = 1$. Let $\hat{p}^i = \frac{1}{2} + \frac{c^i}{2}$ for all $i \in \{1, 2\}$. Then, the optimal prices for a price cycle of length n are*

$$p_t^i = \begin{cases} c^1 + \kappa_t\left[(3/2)(1-c^1) + (1/2)(c^2 - c^1)\right], & \text{for } i = 1 \\ \hat{p}^2 + \frac{\kappa_t}{2}\left[(3/2)(1-c^1) + (1/2)(c^2 - c^1)\right], & \text{for } i = 2, \end{cases} \tag{B5}$$

where $\kappa_t = \frac{2(1+\theta_t)}{7}$, $\theta_t = \frac{\beta_{n-t}\theta_{t-1} - 1}{\beta_{n-t+1}}$ for $t = 1 \ldots n$, $\theta_0 = 1$, and $\{\beta_t\}_{t=1}^n$ is a sequence such that $\beta_0 = 1$, $\beta_1 = \frac{7}{4\alpha} + 2$, $\beta_t = \left(\frac{7}{4\alpha} + 2\right)\beta_{t-1} - \beta_{t-2}$ for $t = 2 \ldots n-1$ and $\beta_n = \frac{7}{4\alpha}\beta_{n-1} - \beta_{n-2}$.

Proof of Lemma 4 Next, we characterise the optimal n-period non-increasing pricing sequence. Let $\boldsymbol{p}$ be the pricing sequence of length n, i.e. $p_1^i \geq p_2^i \geq \ldots \geq p_n^i$ for all $i \in \{1,2\}$, and $p_t^1 \leq p_t^2$ for all $t = 1,\ldots,n$ and $i \in \{1,2\}$. Then profit function can be written as follows:

$$f_n(\boldsymbol{p}) = \sum_{\substack{t=1\ldots n\\ i=1\ldots 2}} (p_t^i - c^i)(1-p_t^i) + \sum_{t=1\ldots n} (p_t^1 - c^1)(p_t^2 - p_t^1) + \sum_{\substack{t=2\ldots n\\ i=1\ldots 2}} \alpha(p_t^1 - c^1)(p_{t-1}^1 - p_t^1)$$

Let $\hat{p}^i$ denote the non-interaction optimal for product i, i.e. $\hat{p}^i = \frac{1+c^i}{2}$. Then, by letting $p_t^i = \hat{p}^i(1+\delta_t^i)$, substituting $1+c^i = 2\hat{p}^i$, and subtracting the resulting expression from $f_n(\hat{p})$, we obtain

$$\begin{aligned} f_n(\boldsymbol{p}) - f_n(\hat{p}) = &\sum_{\substack{t=1\ldots n\\ i=1\ldots 2}} -(\hat{p}^i)^2(\delta_t^i)^2 + \sum_{t=1\ldots n}\left[\hat{p}^1\hat{p}^2\delta_t^1\delta_t^2 - (\hat{p}^1)^2(\delta_t^1)^2\right] + \sum_{\substack{t=2\ldots n\\ i=1\ldots 2}}\left[\alpha(\hat{p}^1)^2\delta_{t-1}^1\delta_t^i - \alpha(\hat{p}^1)^2(\delta_t^1)^2\right]\\ &+ \sum_{t=1\ldots n}\left[(\hat{p}^1)^2 m(r^2-1)\delta_t^1 + \hat{p}^1\hat{p}^2 m\delta_t^2\right] + \sum_{i=1\ldots 2}\left[\alpha(\hat{p}^1)^2 m(\delta_1^1 - \delta_n^1)\right] \end{aligned}$$

Dividing by the sequence length, n, we get the average profit per period Δf_n when n-period pricing sequence is used. In matrix form, we can rewrite Δf_n as follows:

$$\Delta f_n = \frac{\sum_{i=1}^2(\hat{p}^1)^2}{n}\left[\frac{1}{2}\delta^T\cdot\Omega\cdot\delta - e^T\cdot\delta\right] \tag{B6}$$

where Ω is $2n\times 2n$ matrix, and e is $2n\times 1$ vector. Let Ω_{ij} be $n\times n$ dimensional submatrix in Ω, where $i,j = 1\ldots 2$. Ω_{ij} can be expressed as follows: $\Omega_{11} = \begin{pmatrix} -2 & \alpha & 0 & \ldots & 0 & 0\\ \alpha & -2(1+\alpha) & \alpha & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & \alpha & -2(1+\alpha)\end{pmatrix}$ $\Omega_{12} = \Omega_{21} = \frac{\hat{p}^2}{2\hat{p}^1}I_{n\times n}$, $\Omega_{22} = -\left(\hat{p}^2/\hat{p}^1\right)^2 I_{n\times n}$, where $I_{n\times n}$ is $n\times n$ dimensional unit matrix. Likewise, e_i can be expressed as follows: $e_1 = \frac{m}{2}(1-r)I_{n\times 1} - m\alpha J_{n\times 1}$ and $e_2 = -\frac{m}{2}\frac{\hat{p}^2}{\hat{p}^1}I_{n\times 1}$, where $I_{n\times 1} = (1,\ldots,1)^T$, $J_{n\times 1} = (-1,0,\ldots,0,1)^T$, $m = 1 - c^1/\hat{p}^1$, and $r = \frac{\hat{p}^2-\hat{p}^1}{\hat{p}^1-c^1}$, where $()^T$ is transpose operator. We can take the derivative of quadratic form and set it to zero to obtain the optimal δ and γ in matrix notation:

$$\Omega = \begin{pmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{pmatrix}\cdot\begin{pmatrix}\delta^1\\ \delta^2\end{pmatrix} = \begin{pmatrix}e_1\\ e_2\end{pmatrix} \tag{B7}$$

Solving δ^2 in terms of δ^1 in (B7), we obtain: $\delta^2 = -\Omega_{22}^{-1}\cdot\Omega_{21}\cdot\delta^1 + \Omega_{22}^{-1}\cdot e_2 \to \delta_t^2 = \frac{\delta_t^1\hat{p}^1}{2\hat{p}^i} + \frac{m\hat{p}^1}{2\hat{p}^i}$. Substituting it back to (B7), we obtain the following equation for δ^1: $\left(\Omega_{11} + \Omega_{12}\frac{\hat{p}^1}{2\hat{p}^i}\right)\cdot\delta^1 = \left(e_1 - \Omega_{12}\frac{m\hat{p}^1}{2\hat{p}^i}\right)$. Dividing both sides by α, we obtain the following matrix equation:

$$\begin{pmatrix} -u & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0\\ 1 & -u-2 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 1 & -u-2 & 1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & -u-2 \end{pmatrix}\cdot\boldsymbol{\delta} = \begin{pmatrix}1-\frac{1}{z}\\ 1\\ \vdots\\ 1\\ \vdots\\ 1+\frac{1}{z}\end{pmatrix} zm \tag{B8}$$

where $\boldsymbol{\delta} = (\delta_1,\ldots,\delta_n)^T$, $u = 2/\alpha - 1/(4\alpha) = \frac{7}{4\alpha}$ and $z = 1/(4\alpha) - r/(2\alpha) = \frac{1}{\alpha}\left[\frac{1}{4} - \frac{r}{2}\right]$. We can rewrite the right hand side as follows:

$$\begin{pmatrix}1-\frac{1}{z}\\ 1\\ \vdots\\ 1\\ \vdots\\ 1+\frac{1}{z}\end{pmatrix} zm = \overbrace{\begin{pmatrix}-1\\ 0\\ \vdots\\ 0\\ \vdots\\ 1\end{pmatrix}\left(1-\frac{z}{u}\right)m}^{\boldsymbol{p}^2} + \overbrace{\begin{pmatrix}1-\frac{1}{u}\\ 1\\ \vdots\\ 1\\ \vdots\\ 1+\frac{1}{u}\end{pmatrix} zm}^{\boldsymbol{p}^2} \tag{B9}$$

In Lemma 5 below, we find γ and ζ such that $\Omega\gamma m = \boldsymbol{p}^1$ and $\Omega\zeta m = \boldsymbol{p}^2$. Hence, $\delta^1 = (\gamma+\zeta)m$ satisfies (B8).

Thus, $\delta_t^1 = (\gamma_t - \frac{z}{u})m$ is the optimal perturbation. Note that we can rewrite $\gamma_t = \gamma_0\theta_t = \theta_t - \frac{z}{u}\theta_t$ where $\theta_t = \frac{\beta_{n-t}\theta_{t-1}-1}{\beta_{n-t+1}}$ for $t = 1\ldots n$, $\theta_0 = 1$. This implies that optimal price of product 1 (cheap product) is

$$p_t^1 = \hat{p}^1(1+\delta_t^1) = c^1 + 2\kappa_t\left[\left(\hat{p}^1 - c^1\right)\frac{3}{2} + \frac{1}{2}\left(\hat{p}^j - \hat{p}^1\right)\right] = c^1 + \kappa_t\left[\left(1-c^1\right)\frac{3}{2} + \frac{1}{2}\left(c^2 - c^1\right)\right]$$

where $\kappa_t = \frac{2(1+\theta_t)}{7}$. Recall that when $i = 2$, $\delta_t^i = \frac{\delta_t^1 \hat{p}^1}{2\hat{p}^i} + \frac{m\hat{p}^1}{2\hat{p}^i}$, and this implies that optimal price for product 2 (expensive product) is

$$p_t^2 = \hat{p}^2(1+\delta_t^2) = \hat{p}^2 + \kappa_t\left[\left(\hat{p}^1 - c^1\right)\frac{3}{2} + \frac{1}{2}\left(\hat{p}^2 - \hat{p}^1\right)\right] = \frac{1+c^2}{2} + \frac{\kappa_t}{2}\left[\left(1-c^1\right)\frac{3}{2} + \frac{1}{2}\left(c^2 - c^1\right)\right]$$

□

LEMMA 5

(i) *Let* $\gamma = \begin{bmatrix}\gamma_1 & \gamma_2 & \dots & \gamma_{n-1} & \gamma_n\end{bmatrix}^T$ *where* $\gamma_t = \frac{\beta_{n-i}\gamma_{t-1} - \gamma_0}{\beta_{n-t+1}}$ *for* $t = 1 \dots n$, $\gamma_0 = (1 - \frac{z}{u})$, $\frac{z}{u} = \frac{\frac{1}{2}-r}{7/2}$ *and* $\{\beta_t\}_{t=1}^n$ *is a sequence such that* $\beta_0 = 1$, $\beta_1 = u+2$, *and*

$$\beta_t = \begin{cases}(u+2)\,\beta_{t-1} - \beta_{t-2} & \text{for } t = 2\dots n-1\\ u\beta_{t-1} - \beta_{t-2} & \text{for } t = n.\end{cases} \tag{B10}$$

Then, γ *satisfies* $\Omega\gamma m = \boldsymbol{p}^1$

(ii) $\zeta = \begin{bmatrix}-\frac{z}{u} & -\frac{z}{u} & \dots & -\frac{z}{u} & -\frac{z}{u}\end{bmatrix}^T$ *satisfies* $\Omega\zeta m = \boldsymbol{p}^2$

Proof of Lemma 5 This result follows by substituting the given γ and ζ into the equations. □

Note that since problem parameters are stationary, the total profit generated within a price cycle depends only on the cycle's length n. Let Π_n be the optimal total profit generated by a price cycle of length n. Then, it is straightforward to show that Π_n increases in n. In order to characterise the optimal length of price cycle, we define $\overline{\Pi}_n = \Pi_n/n$, the average optimal profit per period generated by a price cycle of length n. In the next Lemma, we show that $n^* = 2$ maximises $\overline{\Pi}_n$ for all α and cost parameter values; in other words, a cyclic pricing policy with a cycle length of 2 periods maximises the average profit:

LEMMA 6 *Consider model* $(\boldsymbol{P}_s)$ *with* $q = \infty$ *and* $\beta = 1$. *The average cycle profit,* $\overline{\Pi}_n$, *is maximised by a price cycle of length* $n = 2$ *for all* α *and cost parameters.*

Proof of Lemma 6 If we substitute the optimal $\delta = \Omega^{-1}e$ into quadratic form (B6) to obtain the optimal average profit per period, Δf_n^*:

$$\begin{aligned}\Delta f_n^* &= \frac{\sum_{i=1}^2(\hat{p}^1)^2}{n}\left[\frac{1}{2}e^T\Omega^{-1}\Omega\Omega^{-1}e - e^T\Omega^{-1}e\right] = \frac{\sum_{i=1}^2(\hat{p}^1)^2}{n}\left[\frac{1}{2}e^T\Omega^{-1}e - e^T\Omega^{-1}e\right]\\ &= -\frac{\sum_{i=1}^2(\hat{p}^1)^2}{n}\left[\frac{1}{2}e^T\Omega^{-1}e\right] = -\frac{\sum_{i=1}^2(\hat{p}^1)^2}{2n}\sum_{i=1}^2(e_i^T\delta^i)\end{aligned} \tag{B11}$$

Substituting $\delta^2 = -\Omega_{22}^{-1}\Omega_{21}\delta^1 + \Omega_{22}^{-1}e_i$:

$$\begin{aligned}\Delta f_n^* &= -\frac{\sum_{i=1}^2(\hat{p}^1)^2}{2n}\left[e_1^T\delta^1 - e_2^T\Omega_{22}^{-1}\Omega_{21}\delta^1 + e_2^T\Omega_{22}^{-1}e_2\right]\\ &= -\frac{\sum_{i=1}^2(\hat{p}^1)^2}{2n}\left[\left(e_1^T - e_2^T\Omega_{22}^{-1}\Omega_{21}\right)\delta^1 + e_2^T\Omega_{22}^{-1}e_2\right]\end{aligned} \tag{B12}$$

where

$$\left(e_1^T - e_2^T\Omega_{22}^{-1}\Omega_{21}\right) = m\begin{pmatrix}(1/4)(1/2-r)\\ \vdots\\ (1/4)(1/2-r)\end{pmatrix} + m\begin{pmatrix}-\alpha\\ 0\\ \vdots\\ 0\\ \alpha\end{pmatrix},\; e_2^T\Omega_{22}^{-1}e_2 = \frac{-m^2n}{2}(1/4)$$

Substituting these terms back into (B12),

$$\Delta f_n^* = -\frac{\sum_{i=1}^2(\hat{p}^1)^2}{2n}\left[m(1/4)(1/2-r)\sum_{t=1}^n\delta_t^1 - m\alpha(\delta_1^1 - \delta_n^1) - \frac{m^2n}{2}(1/4)\right]$$

Note that $\sum_{t=1}^n\delta_t^1 = \frac{1}{u}(\delta_1^1 - \delta_n^1) - nmz/u$ due to the the row sum of the matrix equation. Therefore, Δf_n^* can be expressed as follows:

$$\Delta f_n^* = -\frac{\sum_{i=1}^2(\hat{p}^1)^2}{2n}\left[m\left(\frac{1}{u}\frac{1}{2}(1/2-r) - \alpha\right)(\delta_1^1 - \delta_n^1) - \frac{m^2nz}{u}\frac{1}{2}(1/2-r) - \frac{m^2n}{4}\right]$$

Recall that $\delta_1^1 - \delta_n^1 = m(\gamma_1 - \gamma_n)$. Substituting into Δf_n^* and collecting terms that depend on n, we obtain:

$$\begin{aligned}\Delta f_n^* &= \frac{\sum_{i=1}^2(\hat{p}^1)^2}{2}\left[m^2\left(\alpha - \frac{1}{u}\frac{1}{2}(1/2-r)\right)(\gamma_1 - \gamma_n)/n + \frac{m^2}{4} + \frac{m^2z}{2u}(1/2-r)\right]\\ &= A_1(\gamma_1 - \gamma_n)/n + A_2\end{aligned} \tag{B13}$$

where $A_1 = \frac{\alpha(\hat{p}^1-c^1)^2 4}{7}\left(\frac{3}{2}+\frac{1}{2}\frac{\hat{p}^2-\hat{p}^1}{\hat{p}^1-c^1}\right)$ and $A_2 = \frac{(\hat{p}^1-c^1)^2 4}{7}\left(\frac{7}{16}+\left(\frac{1}{2}\left[\frac{1}{2}-\frac{\hat{p}^2-\hat{p}^1}{\hat{p}^1-c^1}\right]\right)^2\right)$. Notice that the increase in the average profit under an optimal non-increasing price sequence of length n depends only on the difference between the price distortion of product 1 (from its noninteraction price, $\hat{p}^1$) in the first period (i.e. γ_1^1) and the price distortion in the last period (i.e. γ_n^1). Both quantities depend on the length of the price sequence, n. In order to show that a price sequence of length 2 is optimal, we develop the upper bound on the average profit increase when a n-period optimal decreasing pricing sequence is used. Using Lemma 9, we characterise asymptotic properties for both γ_1 and γ_n, which will be useful when we prove the optimality of period 2. For any optimal decreasing pricing sequence of length n, the average profit improvement is bounded by:

$$\Delta f_n^* = A_1(\gamma_1-\gamma_n)/n + A_2 \le \Delta F_n^* = \frac{A_1}{n}\left[\gamma_0\left(\frac{1}{r_1-r_\infty}+\frac{1}{r_0-1}\right)\right]+A_2 \tag{B14}$$

It is trivial to show that ΔF_n^* decreases in n, and therefore, following relational structure holds:

$$\begin{array}{ccc} \Delta f_2^* & \le \Delta F_2^* \\ & \vee \\ \Delta f_3^* & \le \Delta F_3^* \\ \vdots & \vdots \\ \Delta f_n^* & \le \Delta F_n^* \\ \vdots & \vdots \end{array} \iff \begin{array}{cc} A_1(\gamma_1-\gamma_2)/2 + A_2 & \le \frac{A_1}{2}\left[\gamma_0\left(\frac{1}{r_1-r_\infty}+\frac{1}{r_0-1}\right)\right]+A_2 \\ & \vee \\ A_1(\gamma_1-\gamma_3)/3 + A_2 & \le \frac{A_1}{3}\left[\gamma_0\left(\frac{1}{r_1-r_\infty}+\frac{1}{r_0-1}\right)\right]+A_2 \\ \vdots & \vdots \\ A_1(\gamma_1-\gamma_n)/n + A_2 & \le \frac{A_1}{n}\left[\gamma_0\left(\frac{1}{r_1-r_\infty}+\frac{1}{r_0-1}\right)\right]+A_2 \\ \vdots & \vdots \end{array} \tag{B15}$$

If we can show that $A_1(\gamma_1-\gamma_2)/2 + A_2 \ge \frac{A_1}{n}\left[\gamma_0\left(\frac{1}{r_1-r_\infty}+\frac{1}{r_0-1}\right)\right]+A_2$ for some $\bar{n}$ then the relational diagram above implies that $\Delta f_2^* \ge \Delta f_n^*$ where $n \ge \bar{n}$. After some algebra, we can show that for all $\alpha \in [0,1]$:

$$\begin{aligned} \frac{\gamma_1-\gamma_2}{2} &= \gamma_0\frac{7\alpha}{(7/2)(7/2+4\alpha)-4\alpha^2} \ge \gamma_0\left[\frac{1}{4}\left(\frac{1}{r_1-r_\infty}+\frac{1}{r_0-1}\right)\right] = \gamma_0\left[\frac{1}{4}\left(\frac{2}{u-2+\sqrt{u^2+4u}}+\frac{1}{1+u}\right)\right] \\ &= \gamma_0\left[\frac{1}{4}\left(\frac{3u+\sqrt{u^2+4u}}{(u-2+\sqrt{u^2+4u})(1+u)}\right)\right] = \gamma_0\left[\frac{2\alpha}{4}\left(\frac{3(7/2)+\sqrt{(7/2)^2+8\alpha(7/2)}}{(7/2-4\alpha+\sqrt{(7/2)^2+8\alpha(7/2)})(7/2+2\alpha)}\right)\right] \end{aligned}$$

Hence $\Delta f_2^* \ge \Delta F_4^* \ge \Delta f_n^*$ where $n \ge 4$. There remains to prove that $\Delta f_2^* \ge \Delta f_3^*$. In fact, we can easily show that for all $\alpha \in [0,1]$:

$$\frac{\gamma_1-\gamma_2}{2} = \gamma_0\frac{\theta_1-\theta_2}{2} = \gamma_0\frac{2\alpha(7/2)}{(7/2)(7/2+4\alpha)-4\alpha^2} \ge \gamma_0\frac{\theta_1-\theta_3}{3} = \gamma_0\frac{(\frac{4}{3})\alpha(7/2)(7/2+6\alpha)}{(7/2)^2(7/2+8\alpha)+8\alpha^2(7/2-2\alpha)}$$

which implies that $\Delta f_2^* \ge \Delta f_3^*$. This completes the proof. □

Lemma 7 and 8 are used in the proof of Lemma 9. Due to space constraints, we provide them without proof[11]:

LEMMA 7 *Let* $r_0 = 2+u$ *For all* $n \ge 1$, $\beta_{i-1}(n)\beta_i(n) \ge r_0^i,\ i=1,\ldots,n$

LEMMA 8 $\lim_{n\to\infty}\frac{\beta_{n-1}(n)}{\beta_n(n)} = \frac{1}{r_1-r_\infty}$ *where* $r_\infty = \frac{u+2-\sqrt{u^2+4u}}{2}$ *and* $r_1 = u$.

LEMMA 9

(i) $\gamma_1(n)$ *is increasing in* n *and* $\lim_{n\to\infty}\gamma_1(n) = \frac{\gamma_0}{r_1-r_\infty}$ *and*

(ii) $\liminf_{n\to\infty}\gamma_n(n) \ge \frac{-\gamma_0}{r_0-1}$ *for all* $n \ge 1$ *where* $r_\infty = \frac{u+2-\sqrt{u^2+4u}}{2}$, $r_1 = u$, *and* $r_0 = 2+u$.

Proof of Lemma 9 From Lemma 8

$$\gamma_1(n) = \frac{\beta_{n-1}(n)\gamma_0-\gamma_0}{\beta_n(n)}.$$

It can be shown that $\gamma_1(n)$ increases as n increases using the fact that both $\frac{\beta_{n-1}(n)}{\beta_n(n)}$ and $\beta_n(n)$ are monotone increasing. Furthermore, the convergence of a sequence, $\frac{\beta_{n-1}(n)}{\beta_n(n)}$ and the fact that $\lim_{n\to\infty}\beta_n(n) = \infty$ imply the existence of the limit. Replacing γ_0 with a as in Lemma 8, we have: $\lim_{n\to\infty}\gamma_1(n) = \lim_{n\to\infty}\frac{\beta_{n-1}(n)}{\beta_n(n)} = \frac{\gamma_0}{r_1-r_\infty}$. To show the inequality, we consider the following system of difference equations: $\gamma_1(n) = \frac{\beta_{n-1}(n)}{\beta_n(n)}\gamma_0 - \frac{1}{\beta_n(n)}\gamma_0$, $\gamma_2(n) = \frac{\beta_{n-2}(n)}{\beta_{n-1}(n)}\frac{\beta_{n-1}(n)}{\beta_n(n)}\gamma_0 - \frac{\beta_{n-2}(n)}{\beta_{n-1}(n)}\frac{1}{\beta_n(n)}\gamma_0 - \frac{1}{\beta_{n-1}(n)}\gamma_0, \ldots, \gamma_n(n) = \frac{\beta_0(n)}{\beta_n(n)}\gamma_0 - \gamma_0\left(\frac{\beta_0(n)}{\beta_{n-1}(n)\beta_n(n)} + \frac{\beta_0(n)}{\beta_{n-2}(n)\beta_{n-1}(n)} + \ldots + \frac{\beta_0(n)}{\beta_1(n)\beta_2(n)} + \frac{\beta_0(n)}{\beta_1(n)}\right)$. It is easy to see that $\gamma_n(n)$ is decreasing in n since the positive term in the above expression decreases while the negative term grows in magnitude as n increases. To bound $\gamma_n(n)$, we utilise the $\beta_i(n)\beta_{i-1}(n)$

terms. Lemma 7 implies that $\beta_i(n)\beta_{i-1}(n)$ is bounded below by r^i and substituting the corresponding terms yields:

$$\begin{aligned}\gamma_n(n) &= \frac{\beta_0(n)}{\beta_n(n)}\gamma_0 - \gamma_0\left(\frac{\beta_0(n)}{\beta_{n-1}(n)\beta_n(n)} + \frac{\beta_0(n)}{\beta_{n-2}(n)\beta_{n-1}(n)} + \ldots + \frac{\beta_0(n)}{\beta_1(n)\beta_2(n)} + \frac{\beta_0(n)}{\beta_1(n)}\right)\\ &\geq \frac{\beta_0(n)}{\beta_n(n)}\gamma_0 - \gamma_0\left(\frac{\beta_0(n)}{r_0^n} + \frac{\beta_0(n)}{r_0^{n-1}} + \ldots + \frac{\beta_0(n)}{r_0^2} + \frac{\beta_0(n)}{r_0}\right) \geq -\gamma_0\beta_0(n)\left(\frac{1}{r_0^n} + \frac{1}{r_0^{n-1}} + \ldots + \frac{1}{r_0^2} + \frac{1}{r_0}\right)\\ &\geq -\gamma_0\beta_0(n)\left(\frac{1}{r_0 - 1}\right) = \frac{-\gamma_0}{r_0 - 1}\end{aligned}$$

The second last inequality holds because $\frac{\beta_0(n)}{\beta_n(n)}\gamma_0$ is nonnegative and the last equality results from from $\beta_0(n) = 1$. □

In a finite horizon problem instance, repeating this one-period high?one-period low pricing strategy is optimal when T is even. On the other hand, such a strategy would leave the last period uncovered when T is odd; therefore, we fine-tune the optimal pricing strategy for the final three periods.

LEMMA 10 *Consider model ($\boldsymbol{P}_s$) with $q = \infty$ and $\beta = 1$. Then, for all α and cost parameter values,*

(1) *If the planning horizon length, T, is an even number, then optimal product prices consist of 2-period price cycles and optimal total profit is equal to $T/2 \times \overline{\Pi}_2$.*
(2) *On the other hand, if T is odd, then optimal product prices consist of two-period price cycles for the first $T - 3$ periods, and a three-period price cycle for the last three periods, and optimal total profit is equal to $(T - 1)/2 \times \overline{\Pi}_2 + \overline{\Pi}_3$.*

Proof of Lemma 10 We proved in Lemma 6 that the optimal profit increase results from repeating the two-period optimal pricing sequence, which implies that if the planning period T is an even number then the pricing plan is involves repeating the two-period optimal pricing sequence $\frac{T}{2}$ times, where

$$p_t^i = \begin{cases} c^1 + \kappa_t\left[(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)\right] & \text{for } i = 1\\ \hat{p}^i + \frac{\kappa_t}{2}\left[(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)\right] & \text{for } i = 2.\end{cases}$$

for $t = hi, lo$ and $\kappa_{hi} = \frac{2(7/2+6\alpha)}{7(7/2+4\alpha)-8\alpha^2}$, $\kappa_{lo} = \frac{2(7/2+2\alpha)}{7(7/2+4\alpha)-8\alpha^2}$.

However, when T is odd, a small modification of the pricing strategy is required for the final three periods. Note that there are two ways to construct a three-period pricing plan. The first is to use (p_{hi}^i, p_{lo}^i) for the first two periods, solve a single period profit maximisation problem that takes into account only the substitution effect, and use the optimal single period prices p_{single}^i for the last period. The second way is to use a three-period decreasing pricing sequence (p_1^i, p_2^i, p_3^i) for the final three periods, where

$$p_t^i = \begin{cases} c^1 + \kappa_t\left[\left(\hat{p}^1 - c^1\right)(3/2) + (1/2)\left(\hat{p}^2 - \hat{p}^1\right)\right] & \text{for } i = 1\\ \hat{p}^i + \frac{\kappa_t}{2}\left[\left(\hat{p}^1 - c^1\right)(3/2) + (1/2)\left(\hat{p}^2 - \hat{p}^1\right)\right] & \text{for } i = 2.\end{cases}$$

for all $t = 1, 2, 3$ and $\kappa_1 = \frac{2(3/2+6\alpha)(3/2+4\alpha)}{(3/2)^2(3/2+8\alpha)+8\alpha(3/2-2\alpha)}$, $\kappa_2 = \frac{2((3/2+8\alpha)(3/2)+8\alpha^2)}{(3/2)^2(3/2+8\alpha)+8\alpha(3/2-2\alpha)}$, and $\kappa_3 = \frac{2((3/2)(3/2+6\alpha))}{(3/2)^2(3/2+8\alpha)+8\alpha(3/2-2\alpha)}$.

The optimal pricing plan for the last three periods is determined by whichever of these two strategies generate the most profit. First, we calculate the profit of $(p_{\text{hi}}^i, p_{\text{lo}}^i, p_{\text{single}}^i)$. Recall that difference between base profit and profit generated by (p_{hi}^i, p_{lo}^i) is equal to $2\Delta f_2 = A_1(\gamma_1 - \gamma_2) + 2A_2$. To calculate the profit in the last period, we need to compute p_{single}^i. We provide the closed form expression for p_{single}^i below:

$$p_{\text{single}}^i = \begin{cases} c^1 + \frac{4}{7}\left[(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)\right] & \text{for } i = 1\\ \hat{p}^i + \frac{2}{7}\left[(\hat{p}^1 - c^1)(3/2) + (1/2)(\hat{p}^2 - \hat{p}^1)\right] & \text{for } i = 2.\end{cases}$$

The difference between base profit and profit generated by p_{single}^i, Δf_1 is:

$$\Delta f_1 = \frac{(\hat{p}^1 - c^1)^2 2}{7}\left(\frac{7}{16} + \left((1/2)\left[\frac{1}{2} - \frac{\hat{p}^2 - \hat{p}^1}{\hat{p}^1 - c^1}\right]\right)^2\right).$$

Note that $\Delta f_1 = A_2$. Hence total increase in the profit generated by $(p_{\text{hi}}^i, p_{lo}^i, p_{single}^i)$ is equal to $A_1(\gamma_1 - \gamma_2) + 3A_2$. The second way of constructing the 3-period plan is (p_1^i, p_2^i, p_3^i), which leads to a total increase in profit by $A_1(\gamma_1 - \gamma_3) + 3A_2$. Therefore, it is enough to show that $A_1(\gamma_1(2) - \gamma_2(2)) + 3A_2 \leq A_1(\gamma_1(3) - \gamma_3(3)) + 3A_2$. But this is true, since $\gamma_1(n)$ increases in n and $\gamma_n(n)$ decreases in n by Lemma 9. □

B.4 *Proof of Proposition 3*

In Propositions 1 and 2, we obtain closed-form expressions for the optimal one-period high?one-period low pricing strategy for model ($\mathbf{P_S}$) when $\beta = 0$ and $\beta = 1$, respectively. Using these expressions, we can express the absolute difference between high and low prices for products i^* and i as follows:

$$\Delta_i^A(\beta = 0) = \begin{cases} (\hat{p}^1 - c^1)\frac{4\alpha}{4\alpha+4-\alpha^2} & \text{if } i = 1 \\ (\hat{p}^2 - c^2)\frac{4\alpha}{4\alpha+4-\alpha^2} & \text{if } i = 2 \end{cases}$$

and

$$\Delta_i^A(\beta = 1) = \begin{cases} \frac{8\alpha}{(7/2)(7/2+4\alpha)-4\alpha^2}\left[(\hat{p}^1 - c^1)(3/2) + (\hat{p}^2 - \hat{p}^1)(1/2)\right] & \text{if } i = 1 \\ \frac{4\alpha}{(7/2)(7/2+4\alpha)-4\alpha^2}\left[(\hat{p}^1 - c^1)(3/2) + (\hat{p}^2 - \hat{p}^1)(1/2)\right] & \text{if } i = 2 \end{cases}$$

where $\hat{p}^i = \frac{1}{2} + \frac{c^i}{2}$. Since $8\alpha \geq 4\alpha$, it is straightforward to show that

$$\Delta_1^A(\beta = 1) \geq \Delta_2^A(\beta = 1).$$

We can rewrite $(\hat{p}^2 - c^2)$ as follows:

$$\hat{p}^i - c^i = \frac{1 + c^i}{2} - c^i = \frac{1 - c^i}{2}$$

Recalling that we assume $c^1 \leq c^2$, we can show that:

$$\hat{p}^1 - c^1 \leq \hat{p}^2 - c^2.$$

This implies that:

$$\Delta_1^A(\beta = 0) \leq \Delta_2^A(\beta = 0)$$

Now, we compare the relative difference between high and low prices for products 1 and 2. Relative difference for product 2 can be written as follows:

$$\Delta_i^R(\beta = 0) = \frac{(\hat{p}^i - c^i)\frac{4\alpha}{4\alpha+4-\alpha^2}}{\hat{p}^i + (\hat{p}^i - c^i)\frac{\alpha(\alpha+2)}{4\alpha+4-\alpha^2}} = \frac{\frac{\hat{p}^i - c^i}{\hat{p}^i}\frac{4\alpha}{4\alpha+4-\alpha^2}}{1 + \frac{\hat{p}^i - c^i}{\hat{p}^i}\frac{\alpha(\alpha+2)}{4\alpha+4-\alpha^2}} = \frac{\frac{1-c^i}{1+c^i}\frac{4\alpha}{4\alpha+4-\alpha^2}}{1 + \frac{1-c^i}{1+c^i}\frac{\alpha(\alpha+2)}{4\alpha+4-\alpha^2}} = \frac{\frac{4\alpha}{\alpha(\alpha+2)}\frac{1-c^i}{1+c^i}}{\frac{4\alpha+4-\alpha^2}{\alpha(\alpha+2)} + \frac{1-c^i}{1+c^i}}$$

Also, using the fact that $\frac{4\alpha+4-\alpha^2}{\alpha(\alpha+2)} > 1$ for all $0 \leq \alpha \leq 1$, we can show that $\Delta_1^R(\beta = 0) \leq \Delta_2^R(\beta = 0)$. Lastly, we show that $\Delta_1^R(\beta = 1) \geq \Delta_2^R(\beta = 1)$. Note that $\Delta_1^R(\beta = 1)$, and $\Delta_2^R(\beta = 1)$ can be written as follows:

$$\Delta_1^R(\beta = 1) = \frac{\Delta_1^A(\beta = 1)}{p_{hi}^1} \text{ and } \Delta_2^R(\beta = 1) = \frac{\Delta_2^A(\beta = 1)}{p_{hi}^2}$$

Using the facts that $p_{hi}^1 \leq p_{hi}^2$ and $\Delta_1^A(\beta = 1) \geq \Delta_2^A(\beta = 1)$, we can show that $\Delta_1^R(\beta = 1) \geq \Delta_2^R(\beta = 1)$. □

B.5 *Proof of Proposition 4*

We consider each case separately.

(1) $\beta = 0$ case: First, we show that an *alternating* one-period high and one -period low pricing strategy generates more profit than a *simultaneous* pricing strategy. Note that products are symmetric (i.e. demand and cost parameters are the same) and cost and demand parameters are stationary (i.e. they do not depend on time). This implies that if simultaneous pricing strategy were adopted, high and low prices would be offered at the same time. This would lead to low demand when high prices are offered, and high demand when low prices are offered. If the capacity at each period, i.e. q, is less than the level of the high demand, some capacity will be unused. This reduces the capacity utilisation in high period. However, we can avoid this completely by offering high and low prices in an alternating fashion. Then, the demand in each period will be constant, thus completely eliminate completely the unused portion of capacity.
Using the stationary and symmetric setting of the problem, we can decompose problem (4) into independent subproblems for each product:

$$\max_{1 \geq p_{hi}^i \geq p_{lo}^i \geq 0} \{f(p_{hi}^i, p_{lo}^i) \text{ s. t. } d_{hi}^i(p_{hi}^i, p_{lo}^i) + d_{lo}^i(p_{hi}^i, p_{lo}^i) \leq q\}$$

where $d_{hi}^i(p_{hi}^i, p_{lo}^i) = (1 - p_{hi}^i)$, $d_{lo}^i(p_{hi}^i, p_{lo}^i) = (1 - p_{lo}^i) + \alpha(p_{hi}^i - p_{lo}^i)$, and $f(p_{hi}^i, p_{lo}^i) = (p_{hi}^i - c)d_{hi}^i(p_{hi}^i, p_{lo}^i) + (p_{lo}^i - c)d_{lo}^i(p_{hi}^i, p_{lo}^i)$. Let $\lambda \geq 0$ be a lagrangian multiplier for constraint $d_{hi}^i(p_{hi}^i, p_{lo}^i) + d_{lo}^i(p_{hi}^i, p_{lo}^i) \leq q$. Let $\boldsymbol{p}^i = (p_{hi}^i, p_{lo}^i)$. Since the above model is a concave optimisation problem with linear constraints, the optimal solution $(\boldsymbol{p}^*)$ must satisfy the following KKT conditions: $\{\nabla f(\boldsymbol{p}^i) = \lambda(\nabla d_{hi}^i(\boldsymbol{p}^i) + \nabla d_{lo}^i(\boldsymbol{p}^i)), d_{hi}^i(\boldsymbol{p}^i) + d_{lo}^i(\boldsymbol{p}^i) \leq q, \lambda(d_{hi}^i(\boldsymbol{p}^i) + d_{lo}^i(\boldsymbol{p}^i) - q) = 0, p_{hi}^i \geq p_{lo}^i\}$. Depending on whether $\lambda = 0$ or $\lambda > 0$, we have two cases:

Case 1: ($\lambda = 0$) Note that in this case, the first KKT condition transforms into $\nabla f(\boldsymbol{p}^i) = 0$. The following prices satisfy this condition:

$$p_t^i = 1 - \frac{1}{4}\frac{4(1-c)(1+\alpha)}{4+4\alpha-\alpha^2}\begin{cases} 2-\alpha & \text{for } t = hi \\ \frac{2+3\alpha-\alpha^2}{1+\alpha} & \text{for } t = lo. \end{cases} \tag{B16}$$

In order to meet capacity constraints in each period, i.e. $d_{hi}^i(\boldsymbol{p}^i) + d_{lo}^i(\boldsymbol{p}^i) \le q$, q must satisfy $q \ge \bar{q} = \frac{4(1-c)(1+\alpha)}{4+4\alpha-\alpha^2}$. Therefore, as long as $q \ge \bar{q}$, the prices in Equations (B16) satisfy all KKT conditions, and hence are optimal solution.
Case 2: ($\lambda > 0$) In this case, the KKT conditions consist of $\{\nabla f(\boldsymbol{p}) = \lambda(\nabla d_{hi}^i(\boldsymbol{p}^i) + \nabla d_{lo}^i(\boldsymbol{p}^i)), d_{hi}^i(\boldsymbol{p}^i) + d_{lo}^i(\boldsymbol{p}^i) = q, p_{hi}^i \ge p_{lo}^i\}$. The prices that satisfy these equations are as follows:

$$p_t^i = 1 - \frac{1}{4}q\begin{cases} 2-\alpha & \text{for } t = hi \\ \frac{2+3\alpha-\alpha^2}{1+\alpha} & \text{for } t = lo. \end{cases} \tag{B17}$$

Note that $p_{hi}^i \ge p_{lo}^i$. It is easy to check that as long as $q < \bar{q}$, $\lambda > 0$. Therefore, as long as $q < \bar{q}$, the prices in Equations (B17) satisfy all KKT conditions, and hence are optimal solution.

(2) $\beta = 1$ case: Before we start the analysis, we show that when $\beta = 1$, the *simultaneous* one -period high and one-period low pricing policy leads to higher profit than the *alternating* policy. Since we assume that $c^1 = c^2 = c$, without loss of generality, we can pick any product and charge the lowest price for it throughout the planning horizon. Let $i = 1$ be the index for lowest priced product. Hence, using Lemma 1, we can express product 2's optimal price as follows:

$$p_t^2 = \frac{1+p_t^1}{2} + \frac{c^2-c^1}{2} = \frac{1+p_t^1}{2}$$

where $t \in$ {hi,lo}. Using this equation, we can eliminate price variables for product 2 (i.e. p_t^2) in Model (4), and reduce the problem to one with 2 decision variables, i.e. the high and low prices of product 1:

$$\max_{1 \ge p_{hi}^1 \ge p_{lo}^1 \ge 0} \{f(p_{hi}^1, p_{lo}^1) \text{ s. t. } d_{hi}(p_{hi}^1, p_{lo}^1) + d_{lo}(p_{hi}^1, p_{lo}^1) \le 2q, d_{lo}(p_{hi}^1, p_{lo}^1) \le q, p_t^2 \ge p_t^1\}$$

where $d_{hi}(p_{hi}^1, p_{lo}^1)$ and $d_{lo}(p_{hi}^1, p_{lo}^1)$ are total demands in the high and low price periods, respectively, i.e.

$$d_{hi}(p_{hi}^1, p_{lo}^1) = 2 - 2p_{hi}^1 \text{ and } d_{lo}(p_{hi}^1, p_{lo}^1) = 2 + 2\left(\alpha p_{hi}^1 - (1+\alpha)p_{lo}^1\right)$$

and $f(p_{hi}^1, p_{lo}^1)$ is profit before inventory costs, i.e.

$$f(p_{hi}^1, p_{lo}^1) = \sum_{t\in\{hi,lo\}}\left(2(p_t^1 - c)(1-p_t^1)\right) + \sum_{t\in\{hi,lo\}}\frac{1}{4}\left((1-p_t^1)\right)^2 + 2\alpha(p_{lo}^1 - c)(p_{hi}^1 - p_{lo}^1).$$

Note that $p_t^2 \ge p_t^1$ is automatically satisfied as long as $0 \le p_t^1 \le 1$ since $p_t^2 = \frac{1}{2} + \frac{1}{2}p_t^1$. Therefore, we can drop $p_t^2 \ge p_t^1$ from the formulation. Also, in our analysis, we first assume that $0 \le p_t^1 \le 1$ and afterwards show that this is satisfied by the optimal prices.
We continue by solving the following problem:

$$\max_{p_{hi}^1 \ge p_{lo}^1} \{f(p_{hi}^1, p_{lo}^1) \text{ s. t. } d_{hi}(p_{hi}^1, p_{lo}^1) + d_{lo}(p_{hi}^1, p_{lo}^1) \le 2q, d_{lo}(p_{hi}^1, p_{lo}^1) \le q\} \tag{B18}$$

Let $\lambda_1 \ge 0$ and $\lambda_2 \ge 0$ be the lagrangian multipliers of constraints 1 and 2, respectively. Let $\boldsymbol{p} = (p_{hi}^1, p_{lo}^1)$. Since Model (B18) is a concave optimisation problem with linear constraints, the optimal solution ($\boldsymbol{p}^*$) has to satisfy the following KKT conditions: $\{\nabla f(\boldsymbol{p}) = \lambda_1(\nabla d_{hi}(\boldsymbol{p}) + \nabla d_{lo}(\boldsymbol{p})) + \lambda_2 \nabla d_{lo}(\boldsymbol{p}), d_{lo}(\boldsymbol{p}) \le q, \lambda_2(d_{lo}(\boldsymbol{p}) - q) = 0, d_{hi}(\boldsymbol{p}) + d_{lo}(\boldsymbol{p}) \le 2q, \lambda_1(d_{hi}(\boldsymbol{p}) + d_{lo}(\boldsymbol{p}) - 2q) = 0, p_{hi}^1 \ge p_{lo}^1\}$. Depending on whether $\lambda_1 = 0$ or $\lambda_1 > 0$ and $\lambda_2 = 0$ or $\lambda_2 > 0$, we have $2 \times 2 = 4$ cases:

Case 1: ($\lambda_1 = 0, \lambda_2 = 0$) Note that in this case, the first KKT condition transform into $\nabla f(\boldsymbol{p}) = 0$. The following prices satisfy this condition:

$$p_t^1 = \begin{cases} 1 - (\hat{p} - c)\frac{16+10\alpha-8\alpha^2}{49/4+14\alpha-4\alpha^2} & t = hi \\ 1 - (\hat{p} - c)\frac{16+22\alpha-8\alpha^2}{49/4+14\alpha-4\alpha^2} & t = lo. \end{cases} \tag{B19}$$

In order to meet the capacity constraint in the second period, i.e. $d_{lo}(\boldsymbol{p}) \le q$, q must satisfy $q \ge q_2 = 8(1-c)\frac{7+11\alpha-2\alpha^2}{49+56\alpha-16\alpha^2}$. Note that the prices above satisfy $p_{hi}^1 \ge p_{lo}^1$. This implies that $d_{hi}(\boldsymbol{p}) \le d_{lo}(\boldsymbol{p})$, and hence $d_{hi}(\boldsymbol{p}) + d_{lo}(\boldsymbol{p}) \le 2q$ automatically holds. Therefore, the prices in equations (B19) satisfy all KKT conditions, and thus are the optimal solution to Model (B18) when $q \ge q_2$.

Case 2: ($\lambda_1 > 0, \lambda_2 = 0$) In this case, the KKT conditions consist of $\{\nabla f(\boldsymbol{p}) = \lambda_1(\nabla d_{hi}(\boldsymbol{p}) + \nabla d_{lo}(\boldsymbol{p})), d_{hi}(\boldsymbol{p}) + d_{lo}(\boldsymbol{p}) = 2q, d_{lo}(\boldsymbol{p}) \le q, p^1_{hi} \ge p^1_{lo}\}$. The prices that satisfy these equations are as follows:

$$p^1_t = \begin{cases} 1 - q\frac{7/2+11/2\alpha-2\alpha^2}{7+8\alpha-\alpha^2} & t = hi \\ 1 - q\frac{7/2+5/2\alpha-2\alpha^2}{7+8\alpha-\alpha^2} & t = lo. \end{cases} \tag{B20}$$

Note that $p^1_{hi} \ge p^1_{lo}$, implying that $d_{hi}(\boldsymbol{p}) \le d_{lo}(\boldsymbol{p})$. Therefore, $d_{lo}(\boldsymbol{p}) \ge q$, which contradicts the KKT conditions. Hence, Case 2 does not lead to a feasible solution.

Case 3: ($\lambda_1 = 0, \lambda_2 > 0$) In this case, the KKT conditions consist of $\{\nabla f(\boldsymbol{p}) = \lambda_2 \nabla d_{lo}(\boldsymbol{p}), d_{lo}(\boldsymbol{p}) = q, d_{hi}(\boldsymbol{p}) + d_{lo}(\boldsymbol{p}) \le 2q, p^1_{hi} \ge p^1_{lo}\}$. The prices that satisfy these equations are as follows:

$$p^1_t = \begin{cases} 1 - (\hat{p} - c)\frac{8+16\alpha+8\alpha^2}{7+14\alpha+6\alpha^2} + q\frac{3/2\alpha+2\alpha^2}{7+14\alpha+6\alpha^2} & t = hi \\ 1 - (\hat{p} - c)\frac{8\alpha+4\alpha^2}{7+14\alpha+6\alpha^2} - q\frac{7/2+7/2\alpha-2\alpha^2}{7+14\alpha+6\alpha^2} & t = lo. \end{cases} \tag{B21}$$

Two KKT inequalities (i.e. $0 \le \lambda_2, d_{hi}(\boldsymbol{p}) + d_{lo}(\boldsymbol{p}) \le 2q$) give rise to two conditions for q: $q \ge q_1 = 8(1-c)\frac{1+\alpha}{7+10\alpha}$, and $q \le q_2 = \frac{8(1-c)(7+11\alpha-2\alpha^2)}{49+56\alpha-16\alpha^2}$. It is easy to see that all the KKT conditions are satisfied when these conditions are met. Hence, the prices in equations (B21) are the optimal solution to Model (B18) when $q \ge q_1$ and $q \le q_2$.

Case 4: ($\lambda_1 > 0, \lambda_2 > 0$) In this case, the KKT conditions consist of $\{\nabla f(\boldsymbol{p}) = \lambda_1(\nabla d_{hi}(\boldsymbol{p}) + \nabla d_{lo}(\boldsymbol{p})) + \lambda_2 \nabla d_{lo}(\boldsymbol{p}), \lambda 2 \le 0, d_{lo}(\boldsymbol{p}) = q, d_{hi}(\boldsymbol{p}) + d_{lo}(\boldsymbol{p}) = 2q, p^1_{hi} \ge p^1_{lo}\}$. The prices that satisfy these equations are as follows:

$$p^1_t = \begin{cases} 1 - q/(2) & t = hi \\ 1 - q/(2) & t = lo. \end{cases} \tag{B22}$$

Note that $p^1_{hi} = p^1_{lo}$. Two KKT conditions (i.e. $\lambda_1 \ge 0, 0 \le \lambda_2$) give rise to two conditions for q: $q \ge 0$, and $q \le q_1 = 8(1-c)\frac{1+\alpha}{7+10\alpha}$. It is easy to check that all KKT conditions are satisfied when these conditions are met. Hence, the prices in Equations (B22) are the optimal solution to Model (B18) when $q \ge 0$ and $q \le q_1$.

Finally, combining Equations (B19), (B21), and (B22), and using $p^2_t = \frac{1}{2} + \frac{1}{2}p^1_t$, we can obtain the complete characterisation for the optimal high and low prices (p^i_{hi} and p^i_{lo}) for products 1 and 2 as summarised in Proposition 4. □

B.6 *Proof of Proposition 5*

We consider each case separately:

(1) $\beta = 0$: From Proposition 4, we note that $p^i_{hi}, i \in \{1,2\}$ and $p^i_{lo}, i \in \{1,2\}$ decrease in q. Note also that rate of decrease of p^i_{hi} with respect to q is $(2-\alpha)$, whereas the rate of decrease of p^i_{lo} is $\frac{2+3\alpha-\alpha^2}{1+\alpha}$. Since $2 - \alpha < \frac{2+3\alpha-\alpha^2}{1+\alpha}$ for $\alpha \in \{0,1\}$, this implies that the difference between $p^i_{hi} - p^i_{lo}$ increases in q for $i \in \{1,2\}$.

(2) $\beta = 1$: Similarly, from Proposition 4, we note that $p^i_{lo}, i \in \{1,2\}$ decreases in q, whereas p^i_{hi} first decreases in q when $q < q_1$, then increases in q when $q_1 \le q < q_2$ and finally stays constant in q afterwards. Also, when $q < q_1$, $p^i_{lo} = p^i_{hi}$, and hence, the depth of the promotion is zero. On the other hand, when $q_1 \le q < q_2$, the depth of the promotion increases in q since p^i_{hi} increases and p^i_{lo} decreases in q. Finally, when $q_2 \le q$, the depth of promotion is constant in q. □

Appendix 3. Computational study: the impact of system parameters

To explore the value of explicitly considering product-specific intertemporal effects and substitution effects when making inventory and pricing decisions, we compare our model in the main text with the others that explicitly consider none, one, or both of these effects. To do this, we define three versions of our model. In other words, these simplified versions ignore either product-specific intertemporal effects, or current substitution effects, or both, and determine what the optimal pricing and ordering policy would be. We then assess the performance of these heuristics in the presence of the product-specific intertemporal effects and current substitution effects that they didn't consider.

C.1 *Design of the computational study*

To keep the number of instances at a manageable size, we assume a base linear demand function $d(p) = a - p$, where $a = 30$ and consider a four-period planning horizon (i.e. $T = 4$). We created a set of experiments by varying problem parameters as follows:

- Capacity Levels: We assume that total capacity in each period is constant and has one of the following three values: (i) Uncapacitated (equivalent to the case where $Q_t = 100$), (ii) Mildly capacitated ($Q_t = 50$), and (iii) Tightly capacitated ($Q_t = 30$) for all $t = 1 \ldots T$.
- Degree of substitution: We consider 3 different values for β: (i) Low ($\beta = \frac{1}{2}$), (ii) Med ($\beta = \frac{3}{4}$), (iii) High ($\beta = 1$)
- Production Costs: We use the following three values for the product costs: (i) Low ($c = 0$), (ii) Medium ($c = 3$), and (iii) High ($c = 6$).

Table C1. Percentage of profit loss due to ignoring temporal and substitution effects.

		$\Delta_{\beta=0}$	$\Delta_{\alpha=0}$	$\Delta_{\alpha\beta=0}$
Capacity	Uncap	13.13%	16.82%	27.45%
	Med	10.69%	12.93%	20.94%
	Tight	24.26%	17.74%	25.11%
	Average	16.03%	15.83%	24.50%
Prod cost	Low	7.86%	8.50%	15.99%
	Med	13.75%	14.57%	23.11%
	High	26.48%	24.43%	34.39%
	Average	16.03%	15.83%	24.50%
β	Low	12.19%	16.04%	20.55%
	Med	15.69%	15.72%	24.13%
	High	20.21%	15.73%	28.81%
	Average	16.03%	15.83%	24.50%

C.2 *Implementation of the computational study*

For each set of parameter values, we compute the optimal policy accounting for both product-specific intertemporal effects and substitution effects and three heuristic policies that ignore either product-specific intertemporal effects, or current substitution effects or both effects. Specifically, for our heuristic policies, we first solve each of the instances with either α set equal to zero for all instances, or β set equal to zero for all instances, or both set equal to zero for all instances. Solving these restricted versions of our model gives us a set of prices. Given these prices, we then evaluate demand given the actual values of α and β for each instance. Since the prices we determined are calculated using our restricted demand functions, we may not have enough capacity to meet this demand. Therefore, we solve an LP to determine the final inventory decisions to optimally allocate capacity given the prices we have determined and meeting demand up to this demand. We call our heuristic policies the $\alpha = 0$ *heuristic*, the $\beta = 0$ *heuristic*, and the $\alpha\beta = 0$ *heuristic*, respectively, and let $h = \{\alpha = 0, \beta = 0, \alpha\beta = 0\}$ be the indices for these policies. We detail each of these heuristic policies below:

Heuristic Policy h

(1) *Let* $\mathrm{P}_h = \{P_t^i\}_{t=1}^T$ *be the heuristic pricing plan found by solving our main model under a restricted demand model* D_h*, where* D_h, $h = \{\alpha = 0, \beta = 0, \alpha\beta = 0\}$ *is obtained by setting, respectively,* $\alpha = 0$*, or* $\beta = 0$ *or both* $\alpha = 0$ *and* $\beta = 0$ *in demand function* D_t^i.

(2) *Calculate the true demand realisation using the true demand function for each period and product generated by the heuristic pricing plan as follows:* $\hat{D}_{h,t}^i = D_t^i(\mathrm{P}_h)$.

(3) *Solve the following linear program to optimally allocate capacity:*

$$\begin{aligned} \Pi_m = \max \sum_{t=1}^{T}\sum_{i=1}^{2}(p_t^i - c_t^i)d_{h,t}^i & \\ \text{subject to} \qquad \sum_{i=1,2} d_t^i \le Q_t & \\ d_{h,t}^i \le \hat{D}_{h,t}^i & \\ d_{h,t}^i \ge 0 \qquad \forall\, t = 1, \ldots, T \ \text{ and } i = 1, 2 & \end{aligned}$$

In the computational study, we compare these policies with respect to their operational performance. To compare the performance of the optimal policy with that of the heuristics, we define percentage improvement in profit for the optimal policy over that of each heuristic h as follows:

$$\Delta_h = \frac{\Pi^* - \Pi_h}{\Pi_h} * 100$$

The results for Δ_h are presented in Table C1.

C.3 *Observations from computational study*

We make the following observations:

- On average, the gain from using the optimal policy with respect to all other heuristic policies is quite significant. The average gain is 18.79%.

- In general, the gain from using the optimal policy decreases as the capacity becomes more constrained. This is intuitive since when the capacity is tight, the firm can only satisfy the current demand. Therefore, ignoring either temporal effect or substitution effect or both does significantly hurt the firm's profits.
- In general, the gain from using the optimal policy increases as product costs increase. This is because both overage and underage values increase as product costs increase, and hence the firm is penalised more by supply–demand mismatches.
- Comparing the performance of the heuristic policies to the optimal policy, we notice that correctly accounting for only substitution or intertemporal effects leads to about significant reduction in profits. On average, focusing only on substitution effects (i.e. following an $\alpha = 0$ policy) leads to approximately a 15.83% reduction in profits, whereas focusing only on temporal effects (i.e. following a $\beta = 0$ policy) leads to a 16.03% reduction. Ignoring both factors leads to an average of a 24.50% decrease in profits.
- These observations imply that it is essential to account for both substitution and intertemporal effects when making pricing decisions. Otherwise, there is a risk of potential losses due to excessive supply–demand mismatches.

Appendix 4. Additional tables for the empirical survey

Table D1. Summary statistics.

						R_{ijkt}		Δ_{ijk}	
Retailer	$A_j\ (m^2)$	Category	A_{jk}	M_{jk}	DS_{jk}	N	Mean (Std. Dev.)	N	Mean (Std. Dev.)
IGA	612	Detergent	3.35	52	52	73	29.51% (6.83%)	84	0.562 (0.166)
		Shampoo	8.53	85	68	42	27.76% (6.47%)	52	0.573 (0.27)
		Total		137	120	115	28.87% (6.72%)	136	0.566 (0.211)
Loblaws	709	Detergent	1.62	40	40	43	16.43% (9.22%)	108	0.292 (0.221)
		Shampoo	9.24	67	58	317	17.66% (12.22%)	157	0.815 (0.318)
		Total		107	98	360	17.51% (11.9%)	265	0.602 (0.382)
Metro	1055	Detergent	5.10	87	80	14	16.58% (1.92%)	67	0.192 (0.265)
		Shampoo	8.48	55	44	50	10.85% (6.31%)	40	0.699 (0.178)
		Total		142	124	64	12.1% (6.12%)	107	0.381 (0.341)
Provigo	1589	Detergent	22.88	157	123	45	17.05% (8.25%)	54	0.584 (0.34)
		Shampoo	33.81	109	74	78	14.63% (10.06%)	87	0.602 (0.258)
		Total		266	197	123	15.52% (9.47%)	141	0.595 (0.291)

Table D2. Correlation coefficients for the model demonstrating Results 1 and 2.

	Mean	Std Dev	1	2	3	4	5	6
1. R_{ijkt}	0.1859	0.1139	1					
2. A_{jk}	15.8005	9.7008	-0.0667	1				
3. A_j	1235.0	410.6	-0.1901***	0.8948***	1			
4. U_{ijkt}	0.5634	0.4963	-0.0354	0.1207**	0.0648	1		
5. DS_{jk}	91.1677	32.3945	-0.0570	0.7467***	0.8551***	0.0242	1	
6. C_k	0.7357	0.4413	-0.1826***	0.0628	0.3817***	-0.0373	0.5308	1

Table D3. Correlation coefficients for the model showing Result 3.

	Mean	Std Dev	1	2	3
1. Δ_{ijk}	0.5565	0.3350	1		
2. DS_{jk}	77.8752	28.0687	0.3520***	1	
3. C_k	0.2096	0.4073	0.471***	0.4847***	1
4. IGA	0.4083	0.4919	0.0153	−0.2936***	−0.1395**
5. $Loblaws$	0.1649	0.3714	0.1120**	0.7451***	0.1243**
6. $Metro$	0.2173	0.4127	−0.2329***	−0.4214***	−0.1279**
7. $Provigo$	0.5177	0.5001	0.0609	−0.2191***	0.1047**