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Pricing and Manufacturing Decisions When Demand Is a Function of Prices in Multiple Periods

Hyun-soo Ahn
Ross School of Business, University of Michigan, Ann Arbor, Michigan 48109, hsahn@umich.edu

Mehmet Gümüş
Desautels Faculty of Management, McGill University, Montréal, Quebec, Canada H3A 1G5, mehmet.gumus@mcgill.ca

Philip Kaminsky
Department of Industrial Engineering and Operations Research, University of California at Berkeley, Berkeley, California 94720, kaminsky@ieor.berkeley.edu

In most deterministic manufacturing decision models, demand is either known or induced by pricing decisions in the period that the demand is experienced. However, in more realistic market scenarios consumers make purchase decisions with respect to price, not only in the current period, but also in past and future periods. We model a joint manufacturing/pricing decision problem, accounting for that portion of demand realized in each period that is induced by the interaction of pricing decisions in the current period and in previous periods. We formulate a mathematical programming model and develop solution techniques. We identify structural properties of our models and develop closed-form solutions and effective heuristics for various special cases of our models. Finally, we conduct extensive computational experiments to quantify the effectiveness of our heuristics and to develop managerial insights.

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1. Introduction

In recent years, as manufacturing and supply chains have become more and more efficient, the conflict between production planning and marketing has become more apparent. For example, in recent discussions with the manager of a bread production plant associated with a major supermarket chain, the manager indicated to us that the most significant factor leading to increased inventory levels and decreased efficiency in his plant is the unpredictability of demand *due to promotional and pricing decisions made by the marketing group.*

As a result, there is a growing research literature focusing on joint marketing decision making and production planning. Its objective is to develop approaches that avoid conflicting marketing and operations planning decisions by integrating marketing/pricing decisions and manufacturing decisions to jointly achieve a common objective. A variety of aspects of joint pricing and manufacturing models have been analyzed in the operations management and marketing literature. The models vary from constant demand, EOQ-like frameworks, to nonstationary discrete and continuous-time frameworks. Some models allow no replenishment during a price planning horizon (appropriate for perishable or fashion goods), while others consider models which allow production or inventory replenishment over time. The models also differ in assumptions on demand and consumer behavior.

A considerable body of work has been developed that considers joint pricing and production models of perishable or seasonal goods—typically referred to as *revenue management*—traditionally applied to the airline, hotel, and car rental industries, and similar techniques (such as mark-down pricing) have been adopted for perishable or seasonal products. Lazear (1986), Gallego and van Ryzin (1994, 1997), Bitran and Mondschein (1997), Subrahmanyan and Shoemaker (1996), Aviv and Pazgal (2005), and Elmaghraby et al. (2006) consider revenue management problems related to our models, and the area is surveyed in review papers by McGill and van Ryzin (1999), Petruzzi and Dada (1999), Bitran and Caldenty (2003), and the textbook by Talluri and van Ryzin (2004). Federgruen and Heching (1999) and Chen and Simchi-Levi (2004a, b) consider period review stochastic production-pricing models, and Eliashberg and Steinberg (1993), Elmaghraby and Keskinocak (2003), Yano and Gilbert (2004), and Chan et al. (2004) provide comprehensive surveys of research milestones and future opportunities for joint production-pricing problems.

To the best of our knowledge, the earliest paper in which price and production quantity are both decision variables is by Whitin (1955), who extends the basic EOQ model to include a revenue term, and finds the optimal price and lot size using a calculus-based approach. Kunreuther and Richard (1971) propose the same model, and compare decentralized decision making to centralized decision making. Abad (1988) extends Whitin's model to include all-unit quantity discounts, and proposes an iterative algorithm that is guaranteed to converge to the global optimal price and lot size. He applies his model to linear and constant price elastic demand curves and presents a numerical study which compares the decentralized solution to the centralized one, similar to Kunreuther and Richard (1971). Due to the nonconcavity of the profit term in EOQ-type analysis, a global solution may require exhaustive enumeration of local optima. Lee (1993) applies a geometric programming framework to extend pricing the EOQ model to the continuous quantity discount case, and obtains a globally optimal solution to problems under the constant-price elastic demand curve assumption. Kim and Lee (1998) consider a pricing EOQ model where capacity can be expanded or reduced at a cost.

There is also a set of papers that considers the joint production and pricing problem as an optimal control problem. Among these, Stokey (1979) considers the problem faced by a monopolist whose marginal cost of production is decreasing over time. Pekelman (1974), Thompson et al. (1984) and Feichtinger and Hartl (1985) analyze optimal control problems that consider convex production cost with linear and nonlinear demand functions and obtain various planning-horizon results. Cohen (1977) considers an EOQ model with exponentially decaying inventory where demand is a deterministic function of price. Rajan et al. (1992) study the same model with dynamic pricing during the planning horizon and Abad (1996) extends the results obtained by Rajan et al. (1992) to the case in which partial backlogging is allowed.

For the deterministic discrete-time production framework, the trend has been to extend Wagner and Whitin's (1958) model to include price at each period as a decision variable. Under the assumption that the demand in a given period is independent of the prices offered in the adjacent periods and that there is no capacity constraint, Thomas (1970) develops an approach to calculate optimal price and production schedule for each of the T periods and obtains planning horizon results that extend the results of Wagner and Whitin (1958). Kunreuther and Schrage (1973) and Gilbert (1999) consider the special case of constant pricing. Chan et al. (2000) analyze the capacitated discrete-time problem without setup costs and extend the greedy resource allocation algorithm of Federgruen and Groenevelt (1986) to their model. Deng and Yano (2006) consider both capacity and setup costs in their discrete-time problem and show that introduction of prices into the capacitated lot-sizing problem does not change the fundamental structure of the optimal production decisions characterized by Florian and Klein (1971). Charnsirisakskul et al. (2006) consider a joint pricing and scheduling problem when the firm can predict the demand as a function of price.

Virtually all this research considers demand at each period to be a function of price in that period and independent of price in other periods. In many cases, customers may consider making a purchase for several periods, so that demand in a period is a function of both price in that period and prices in other periods. Our model explicitly considers these intertemporal demand-price interactions so that realized demand in each period is dependent on realized demands in previous periods as well as past and current prices. Specifically, in each period a potential pool of customers enters the market and these customers remain in the system for more than one period. Some of the customers entering the system in a given period who cannot afford the product in that period have the patience to wait until the price drops to a level they can afford. We first consider a firm's problem facing customers who purchase the product during the first period in which the price falls below their reservation price, then later extend the result to the case with consumers who are aware of a future price pattern during their tenure and buy at the period when their utility is maximized.

Several papers are closely related to our research. In papers by Sorger (1988), Kopalle et al. (1996), Fibich et al. (2003), and Popescu and Wu (2007), authors consider demand models in the dynamic pricing setting, where past prices affect the current demand via reference price formation mechanisms. Their goal is to identify the properties of the optimal pricing path under various assumptions on the functional form of reference price effects on current demand. Bersanko and Winston (1990) consider a markdown pricing mechanism of a monopolist facing a fixed number of customers during a planning horizon. They model a finite horizon problem where customers who do not purchase in each period remain in the market and they identify the properties of optimal markdown pricing sequences when all customers are strategic or myopic. Our paper is most closely related to Conlisk et al. (1984) and Sobel (1991), which consider a durable good monopolist in a market where a stationary cohort of new customers arrives in each period and no customer leaves the market before purchasing the good. After introducing our model in the next section, we compare it to the models in these papers.

In the next section, we introduce our model, in which demand is allowed to remain in the system for more than one period. We also introduce our results and algorithms. Subsequent sections detail these results and the associated proofs and describe our computational analysis.

2. Model and Main Results

We consider a deterministic discrete time T period finite horizon capacitated production model. For each period, decision variables p_t and x_t, $t = 1, 2, \ldots, T$, represent pricing and production decisions, respectively. Production capacity, per unit production cost, and per unit holding cost can vary from period to period, and are represented by Q_t, c_t, and h_t, $t = 1, 2, \ldots, T$. Demand in each period is a function of price, as we discuss in detail below.

Our model captures, in a relatively stylized and tractable way, the impact of pricing decisions in one period on demand in others. We are motivated by our own experience when purchasing items like cars or computers, where we have in the past first decided on a budget and then waited for the price to fall within our budget. This concept does not only apply to expensive goods, however. Some of the authors of this paper wear Jockey brand undershirts, and were also motivated by their experience purchasing this product. Jockey has semiannual sales each year, in which all prices are discounted 25%. One author, the big spender, upon noticing that his undershirts could do with replacing, heads to the local department store and stocks up. Another author, the thrifty one, waits some period of time up to six months and stocks up at 25% off. Readers also may be familiar with the Barilla SpA (A) case, which describes issues faced by the Barilla Pasta company, including Barilla's customers' tendencies to increase purchase levels of dry pasta during promotional periods and decrease purchase levels at other times (see Hammond 1994).

To model this behavior, we divide demand in each period t into *current demand* and *residual demand*. Current demand is that demand generated by customers who enter the system at that period (for example, the big spender in the undershirt example). Residual demand is that portion of demand resulting from customers who entered the system in previous periods, but who have not yet made a purchase (for example, the thrifty author in the period in which Jockey underwear is on sale).

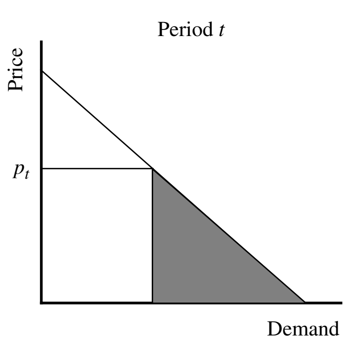
We model current demand in period t using a standard linear price-demand curve:

$$D_t(p_t) = d_t^o - s_t p_t,$$

where d_t^0 is the maximum demand, s_t is the price-demand sensitivity, and

$$p_t \leqslant \frac{d_t^o}{s_t}.$$

Figure 1. The price-demand curve (one period).



In Figure 1, we illustrate this simple demand curve. Note that for a particular price p_t, there is a subset of customers (represented by the shaded portion of the diagram) who would have purchased if the price were lower (in other words, if the price was at or below their reservation price), and who are thus potentially part of the residual demand in future periods.

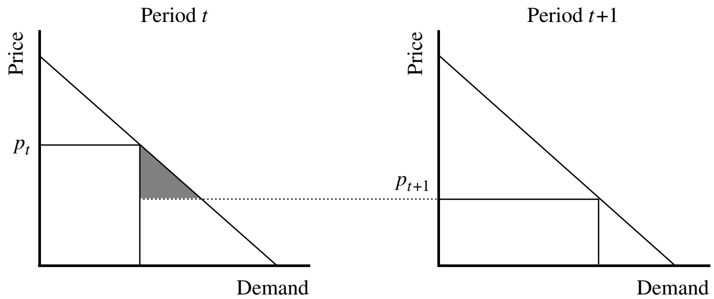
Indeed, in Figure 2 we begin to illustrate the concept of residual demand. The curve on the left represents the current demand curve in period t, as in Figure 1. The curve on the right represents the current demand curve in period $t+1$. Assume that we have set the price in period $t+1$, p_{t+1}, to be lower than the price in period t, p_t. The shaded region on the left represents residual demand from period t realized in period $t+1$—these are customers who found the price too high in period t, but low enough to make purchases in period $t+1$. We define *potential* residual demand from period t to mean demand arriving at period t that can be realized in periods after t if prices are far enough below p_t.

Now, we consider a three-period example, and suppose that prices are ordered so that $p_{t+2} < p_t < p_{t+1}$. We illustrate this example in Figure 3. In this example, no potential residual demand from period t was realized in period $t+1$ because $p_t < p_{t+1}$. However, residual demands from periods t and $t+1$ are realized in period $t+2$ and illustrated with shaded regions in Figure 3.

We further generalize this concept of current and residual demand modeling with two parameters. Potential residual demand from period t does not stay in the system forever; we define parameter K to represent the number of periods that potential demand remains in the system, so that $K = 0$ represents the model with no residual demand, and K is at least 2 in Figure 3. Also, not all of the unmet potential demand in period t remains in the system for K time units—we define α_k^t, $k = 0, 1, \ldots, K$, to represent the proportion of customers who will wait for at least k periods when they join the system at period t such that $1 = \alpha_0^t \geqslant \alpha_1^t \geqslant \alpha_2^t \cdots \geqslant \alpha_K^t \geqslant \alpha_{K+1}^t = 0$ for all t.

There are several limitations to this model. One reasonable critique is that it assumes that consumers are not aware of impending price decreases. Indeed, this model is intended to represent situations in which customers place a high value on a good's availability, and tend to buy it as soon as their budget constraints (i.e., reservation prices) are met. In §7, we briefly consider the situation in which customers are aware of pricing patterns, and in many cases the actual situation may lie between these extremes. Another reasonable critique is our use of linear demand curves; however, this assumption helps us develop models that are amenable to analysis, and helps generate insight which we believe will apply in more complex situations. Also, we note that although our model does not explicitly capture the change in consumers' price sensitivity due to markups and markdowns in prior periods in the same way that typically price/promotion models in the marketing literature

Figure 2. The price-demand curve (two periods).



do, our model is consistent with these models in spirit. Weak demand periods will follow a discount period, and discounts generate revenue from consumers who will not buy at the usual price levels.

Within our framework, we identify several useful properties of the optimal solution and develop algorithms to find good (sometimes optimal) solutions for models with capacity constraints. We believe that our model, while clearly stylized, captures many of the important complexities that exist in real-world dynamic pricing and production problems.

To formally define demand at period t, we first define r_t^k to represent the portion of demand in period t originating from period $t-k$:

$$r_t^k = \begin{cases} d_t^o - s_t p_t & \text{if } k = 0, \\ \alpha_k^{t-k} s_{t-k} \Big[\min\limits_{i \in \{1,\ldots,k\}} p_{t-i} - p_t \Big]^+ & \text{if } 1 \leqslant k \leqslant K. \end{cases} \quad (2.1)$$

We introduce a minimization operator to select the minimum price between periods $t-k$ and $t-1$ because this gives the leftover residual demand after observing the actual prices from periods $t-k$ to $t-1$. Hence, α_k^{t-k} of this leftover residual demand consists of the customers whose reservation prices have not exceeded the product price from period $t-k$ to $t-1$.

Next, we can write demand at period t in terms of r_t^ks as $d_t = \sum_{k=0}^{\min\{t,K\}} r_t^k$. With this demand formulation, we consider the following discrete-time multiperiod production system where at each period we decide both price of the product, p_t, and the production quantity of the product, x_t.

Model 2

$$\max_{p_t, x_t, I_t, d_t, r_t^k} \quad \sum_{t=1}^{T} p_t d_t - \sum_{t=1}^{T} c_t x_t - \sum_{t=1}^{T} h_t I_t \quad (2a)$$

$$\text{s.t.} \quad x_t + I_{t-1} = d_t + I_t, \qquad t = 1, 2, \ldots, T, \quad t_o = 0, \quad (2b)$$

$$x_t \leqslant Q_t, \quad t = 1, 2, \ldots, T, \quad (2c)$$

$$d_t = \sum_{k=0}^{K} r_t^k, \quad t = 1, 2, \ldots, T, \quad (2d)$$

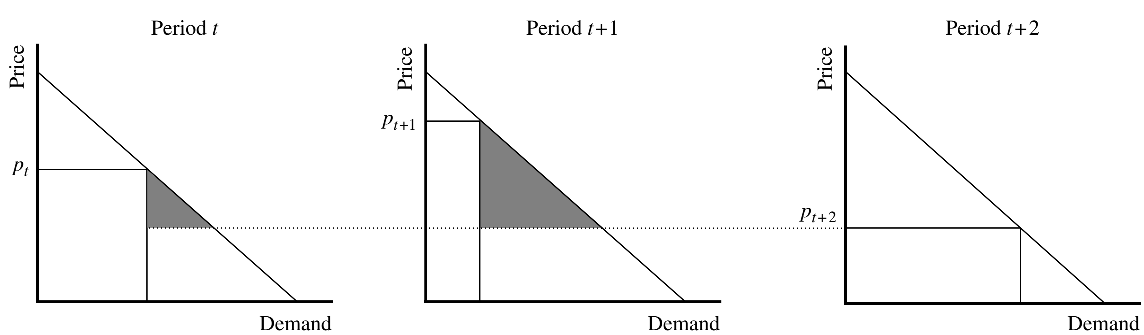
$$0 \leqslant p_t \leqslant d_t^o / s_t, \quad t = 1, 2, \ldots, T, \quad (2e)$$

$$r_t^k = \begin{cases} d_t^o - s_t p_t \quad \text{if } k = 0, \\ \alpha_k^{t-k} s_{t-k} \Big[\min\limits_{i \in \{1,\ldots,k\}} p_{t-i} - p_t \Big]^+ \\ \qquad \text{if } 1 \leqslant k \leqslant \min(K, t-1), \end{cases} \quad (2f)$$

$$x_t \geqslant 0, \quad I_t \geqslant 0, \quad t = 1, 2, \ldots, T. \quad (2g)$$

Our objective is to maximize the net profit (2a) subject to inventory balance (2b), production capacity (2c), and demand realization constraints (2d)–(2f). In addition to the parameters and constraints defined earlier, I_t, $t = 1, 2, \ldots$, represents the inventory at the end of period t. To express constraints (2f) as a set of linear constraints, we introduce additional variables, m_t^k to keep track of the minimum price value observed between periods $t-k$ and $t-1$ and binary variables, y_t^k to indicate whether there is residual demand from period $t-k$ realized in period t (i.e., $y_t^k = 1$ if $r_t^k > 0$

Figure 3. The price-demand curve (three periods).



and zero otherwise). Using y_t^k and m_t^k, we rewrite constraints (2f) as follows:

$$0 \leqslant m_t^k \leqslant p_u \quad \text{for } u = t-k \ldots t-1, \tag{3a}$$

$$0 \leqslant r_t^k \leqslant \alpha_k^{t-k} s_{t-k}(m_t^k - p_t) + (1-y_t^k)\cdot d_{t-k}^o, \tag{3b}$$

$$r_t^k \leqslant y_t^k \cdot d_{t-k}^o$$
$$\text{for } t = 2 \ldots T;\ K = 1 \ldots \min\{t-1, K\}, \tag{3c}$$

$$r_t^0 \geqslant 0, \tag{3d}$$

$$r_t^0 \leqslant d_t^0 - s_t p_t \quad \text{for } t = 1 \ldots T. \tag{3e}$$

It is easy to see that the objective function of Model 2 is neither convex nor concave in general. Note that residual demand is materialized only when past and current prices satisfy a set of conditions, so that the functional form of the objective function changes depending on the relative order of the pricing plan $(p_1, p_2, \ldots, p_T)$. To demonstrate the difficulty posed by this, consider an instance of a two-period problem with identical demand for each period, $\alpha_0^1 = \alpha_0^2 = 1$, $\alpha_1^1 = \alpha$, unlimited capacity, zero holding cost, and zero production cost. This problem is written as:

$$\begin{aligned}
\text{maximize}\quad & R(p_1, p_2) = p_1 d_1(p_1) + p_2 d_2(p_1, p_2)\\
\text{s.t.}\quad & d_1 = d^o - s p_1,\\
& d_2 = (d^o - s p_2) + (\alpha s[p_1 - p_2]^+),\\
& 0 \leqslant p_1 \leqslant d^o/s,\\
& 0 \leqslant p_2 \leqslant d^o/s.
\end{aligned}$$

Substituting expressions for d_1 and d_2 into the objective function reveals that the objective function depends on the relative magnitude of p_1 and p_2:

$$R(p_1, p_2) = p_1(d^o - s p_1) + p_2(d^o - s p_2) + \begin{cases} 0 & \text{if } p_1 \leqslant p_2,\\ \alpha s(p_1 - p_2)p_2 & \text{if } p_1 > p_2. \end{cases} \tag{4}$$

In general, in Model 2 the revenue generated from residual demands depends not only on the current price, but also on the relative order of prices offered for the last K periods, which makes the objective function nonconcave over the feasible region of prices. Typically, such problems are difficult to solve, and an efficient algorithm that finds a global optimal solution does not exist. Below, we characterize the structure of the optimal solution to Model 2. Now, consider Model 2 with the addition of a series of constraints that enforce an ordering of prices. For example, if $T = 3$, these constraints may require $p_3 \geqslant p_1 \geqslant p_2$, or some other ordering. We call these constraints *fixed-ordering constraints*, label them Γ, and prove in online Appendix A (an electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/) that for any such set of constraints Γ—that is, for any fixed ordering of prices—the following result holds:

THEOREM 1. *Model 2, together with the addition of a set of fixed-ordering constraints, is a concave optimization problem.*

To simplify the discussion, we call this mathematical program with the addition of constraints Γ Model 2-Γ with optimal objective function value Z_Γ^*. In §3, we develop a simple algorithm that enumerates possible orderings to solve Model 2, and then use this algorithm to complete a series of computational experiments to assess the effectiveness of our heuristics and to generate managerial insights.

In addition, to develop theoretical insights, as well as efficient optimization algorithms and heuristics, we focus on a variety of special cases of this model. In general, these are cases in which some combination of the following restrictions apply: capacity may be unlimited rather than restricted; residual demand may be limited to one period in the system (that is, $K = 1$); and the model parameters (i.e., demand curves, inventory holding and production costs, proportion of unsatisfied customers who choose to wait, and capacities) may be stationary rather than time varying. We present an algorithm that, although exponential in its complexity, allows us to solve relatively small instances of the original Model 2 presented above, and then move on to special cases of Model 2 to develop solutions and algorithms.

We focus first on the most basic special case of Model 2, the *one-period interaction problem (that is, the problem with $K = 1$) with stationary parameters and unlimited capacity*. For clarity, we refer to this as Model 2-($K = 1$, $Q = \infty$) with stationary parameters. Surprisingly, in this case we can develop a closed-form expression for the optimal pricing policy. In particular, in §4, we develop closed-form expressions for five prices p^{high}, p^{low}, $p^{3\text{-high}}$, $p^{3\text{-medium}}$, and $p^{3\text{-low}}$ and prove:

THEOREM 2. *For any uncapacitated one-period interaction problem with stationary parameters, the following policy is optimal*:

(1) $(p_1, p_2, \ldots, p_{T-1}, p_T) = (p^{high}, p^{low}, \ldots, p^{high}, p^{low})$ *when T is even*;

(2) $(p_1, p_2, \ldots, p_{T-2}, p_{T-1}, p_T) = (p^{high}, p^{low}, \ldots, p^{high}, p^{low}, p^{3\text{-}high}, p^{3\text{-}medium}, p^{3\text{-}low})$ *when T is odd.*

Furthermore, we show that if p^* represents the optimal noninteraction price for this model (that is, the price that maximizes the profit from the demand originated and realized at period t), then p^{high}, $p^{3\text{-high}}$, and $p^{3\text{-medium}}$ are all greater than p^*, and p^{low} and $p^{3\text{-low}}$ are both less than p^*.

These results imply that a periodic pricing strategy that skims off high-valued customers in the first period and serves many of the low-valued customers in the following period is optimal for the infinite horizon one-period interaction problem. This is conceptually similar to some of the results in the intertemporal price discrimination models cited in the introduction to this paper. Under this high-low pricing strategy, it is optimal to price higher than the non-interaction price (p^*) in the first period with the intention of absorbing surplus from high-valued customers, and then price lower than p^* in the next period to absorb surplus from low-valued customers before some of them exit the market.

Returning to our Jockey undershirt example, these results support the strategy of that firm, which manages to extract additional money from the big spending coauthor of this paper, while still selling undershirts to the thrifty coauthor. Of course, many firms, including most famously Wal-Mart, have adopted everyday low price (EDLP) strategies, that appear to contradict these results. In the concluding section of this paper, we address this apparent contradiction.

Next, we identify the following property of solutions to our general Model 2 and use this property to determine optimal solutions for some special cases and to develop heuristic algorithms for others. First, we define:

Definition 1 (Pricing Plan). Let $\mathbf{p} = (p_1, p_2, \ldots, p_t)$ be a vector of prices that is a feasible solution to Model 2. We call $\mathbf{p}$ a *pricing plan*.

Now, define a length K subplan of this plan as follows: $PS(t, K) = (p_{t+1}, \ldots, p_{\min(t+K,T)})$. Note that this is either a length K subsequence of prices (when $t < T - K$), or a sequence that reaches from time $t+1$ until the end of the horizon (when $t \geqslant T - K$). It is easy to see that demand can be decomposed as follows:

Property 1 (Price/Demand Decomposition Property). *For any pricing plan in Model* 2, *if in a length K subplan $PS(t, K)$, $\min(p_{t+1}, \ldots, p_{\min(t+K,T)}) \geqslant p_t$, then there will be no residual demand from periods* 1 *through t impacting demands in any periods $t+1, \ldots, T$.*

Proof. Recall that for the K-period demand interaction case, the customers in the system at period t are those whose reservation prices have not exceeded the price at period t, p_t. Therefore, if the manufacturer prices its product higher than p_t for the next K periods, then the residual customers coming from periods $t-K$ to t will not be able to observe a price that is less than their reservation price, and hence will be forced to leave the system. Therefore, if $\min(p_{t+1}, \ldots, p_{\min(t+K,T)}) \geqslant p_t$ for any t, then customers in the system at time t whose demand is not satisfied by time t will never have their demand satisfied. □

Given this property, we define:

Definition 2 (Regeneration Point). Period t where $2 \leqslant t \leqslant T$ is referred to as a *regeneration point* if $\min(p_t, \ldots, p_{\min(t+K-1,T)}) \geqslant p_{t-1}$.

Furthermore, we assume that the first period of the planning horizon is a regeneration point, and we define dummy period $T+1$ to be another regeneration point.

Definition 3 (Pricing Sequence). Let S_{uv} represent a subset of a pricing plan $\mathbf{p}$ that includes the components of $\mathbf{p}$ for all periods between the two consecutive regeneration points u and v, i.e.,

$$S_{uv} = \{p_i, i = u, \ldots, v-1 \mid u \text{ and } v \text{ are regeneration points}\},$$

where $1 \leqslant u \leqslant v \leqslant T+1$. Then, we call S_{uv} a *pricing sequence*.

If a pricing plan involves n regeneration points, then the plan can be decomposed into $n-1$ pricing sequences. Also note that there must be at least one pricing sequence, which can be the entire pricing plan, and that we define a minimal length pricing sequence to be a pricing sequence that cannot be decomposed into smaller pricing sequences.

Given these definitions, we consider the *one-period interaction problem with time-varying parameters and unlimited capacity*, that is, Model 2-($K = 1$, $Q = \infty$) with time-varying parameters. By characterizing the structure of the optimal solution to this problem, we are able to develop a polynomial time algorithm to optimally solve it. In particular, we observe that for the one-period interaction case, the regeneration point conditions become equivalent to price markups. In other words, if $p_{t-1} \leqslant p_t$, then a regeneration point is said to occur at period t. This simple characterization of the regeneration points leads to an important observation about the structure of pricing sequences in this case:

Lemma 1. *For the one-period interaction case, prices in any minimal length pricing sequence are nonincreasing.*

Proof. Price levels in a minimal length pricing sequence cannot increase, otherwise it would be decomposed into further subsequences which contradicts the minimality of the sequence. □

Based on this lemma, in §4.2 we develop a shortest-path-based optimal algorithm that optimizes over all possible sets of regeneration points.

Unfortunately, when $K > 1$, a result analogous to Lemma 1 does not hold. Consider, for example, the following three-period interaction problem with stationary parameters (that is, $T = 7$, $K = 3$ and $\alpha_k^t = 1$, $D_t = 30 - p_t$, $h_t = 0$, $c_t = 0$ for all $t = 1, \ldots, 7$). The optimal pricing plan ($\mathbf{p}$ = [26.8, 23.6, 18.9, 12.0, 24.5, 18.5, 9.8]) consists of two nonincreasing pricing subplans: (26, 8, 23.6, 18.9, 12.0) and (24.5, 18.5, 9.8). Unfortunately, period 5 is not a regeneration point because customers originating from period 4 purchase the product at period 7. The problem therefore cannot be decomposed into two subproblems, and a sequence of decreasing prices does not coincide with a minimal length price sequence. Thus, the characterization of the optimal price for this problem is in general a formidable task.

However, by focusing on nonincreasing pricing sequences, and using the insight we developed while creating the algorithm and closed-form solution discussed above, in §5 we are able to develop very effective heuristics for the general Model 2. In fact, for the K-period interaction problem with stationary parameters, unlimited capacity, and nonincreasing pricing sequences, we develop a closed-form expression for optimal pricing. For the general Model 2, we develop a shortest-path-based heuristic and show that our heuristic proves to be quite effective in computational study.

As we mentioned in the introductory section, our paper is most closely related to Conlisk et al. (1984) and Sobel (1991), which consider a durable good monopolist in a market where a stationary cohort of new customers arrives in each period and no customer leaves the market before purchasing the good. All customers are strategic (thus, they buy in the period in which their utility is maximized), but customers differ in their willingness to pay—the reservation prices of high-type and low-type customers are V_1 and V_2 ($V_1 > V_2$), respectively. Under this setting, the authors show that a cyclical pricing policy such that the firm sells only to high-type customers in most periods, but offers a markdown (at the end of each cycle) to capture the low-type customers, is a subgame perfect equilibrium.

Our models and results differ from these in several important ways. We allow new customers to enter into the system in each period, but customers will consider buying the good only for a finite number of periods. This assumption plays a crucial role on the length of an optimal pricing sequence and the prices within the sequence. In particular, when $K = 1$ with stationary parameters, we also show that a two-period pricing sequence maximizes the firm's average profit regardless of the spread of reservation prices and that prices are fluctuating around the optimal noninteraction price (i.e., the price which maximizes the profit from the demand originated and realized in the same period). In Conlisk et al. (1984) and Sobel (1991), customers never leave the market until they purchase the good, and thus all customers eventually buy the good. However, the length of an optimal pricing sequence depends on the spread of the reservation prices, and the prices within a sequence are always above (if the proportion of high-type customers is small) or below (if the proportion of high-type customers is large) the optimal noninteraction price. Furthermore, in contrast to previous models, we develop solution techniques for situations in which demand and problem parameters are nonstationary during the planning horizon, which allows us to model situations in which the customers become more (less) price sensitive, and in which the production cost decreases (increases) over time. Finally, we consider both limited and unlimited capacity to investigate how capacity impacts the efficacy of the dynamic pricing strategies. In addition, one focus of this paper is on developing effective solution techniques for these models (closed form or optimal when possible, heuristic when appropriate) so that the impact of intertemporal demand interaction and dynamic pricing can be tested under a variety of conditions.

The remainder of this paper is structured as follows. In §3, we present the algorithm that results from Theorem 1. In §4, we develop the optimal closed-form expression for the one-period interaction problem with stationary parameters and unlimited capacity, and the optimal algorithm for the general one-period interaction problem with unlimited capacity. In §5, we use the insight generated in §§3 and 4 to develop effective heuristics for versions of Model 2. In §6, we present results of computational testing, where we investigate the effectiveness of heuristics for the model, and explore the impact of residual demand on production and pricing decisions. Finally, in §7, we discuss an alternate model and conclude.

3. Optimal Algorithm for Model 2

Recall that Theorem 1 states that Model 2, with the addition of a set Γ of fixed-ordering constraints (that is, Model 2-Γ), is a concave optimization problem. In this section, we use this result to develop an algorithm for the problem.

Because the problem is reduced to an ordinary concave optimization problem under any realization of fixed-ordering constraints Γ, one approach to solve the original problem is to solve Model 2-Γ for all Γ.

Corollary 1. *Let Γ be a set of fixed-ordering constraints determined by each fixed ordering of prices. The optimal pricing and manufacturing plan for Model* 2 (*i.e., the general K-period interaction problem*) *can be found by solving P_Γ for all Γ.*

Algorithm.

Step 1. Let $Z_{best} = 0$.
Step 2. For each Γ
  Solve Model 2-Γ to get the optimal objective function value Z^*_Γ
  If $Z^*_\Gamma > Z_{best}$, then update $Z_{best} = Z^*_\Gamma$
Step 3. $Z^* = Z_{best}$.

Although the algorithm is guaranteed to find the optimal solution for small problems, the number of possible fixed orderings grows exponentially in T. Thus, we are motivated to investigate some structural properties of the optimal solution as well as special cases to design more efficient algorithms. We first start with the one-period interaction problem with unlimited capacity, then use the insights generated in this analysis to solve more complex cases of the problem.

4. Optimal Solutions for the One-Period Interaction Problem with Unlimited Capacity

We first focus on a special case: the one-period interaction problem with unlimited capacity. For clarity, we call this Model 2-($K = 1$, $Q = \infty$). For notational simplicity, we let $\alpha^t_1 = \alpha_t$ for all $t = 1, \ldots, T$. Note that $\alpha^t_k = 0$ for $k > 1$. In the one-period interaction problem, up to $100\alpha_t\%$ of unsatisfied new demand in each period can be satisfied in the next period if the price is sufficiently low. For this special case of our model, we are able to efficiently determine the optimal solution. In particular, we can explicitly solve the problem with stationary parameters, and give a closed-form expression for optimal prices. With time-varying parameters, we develop a shortest-path-based algorithm that determines an optimal pricing plan in polynomial time.

4.1. Model 2-($K = 1$, $Q = \infty$) with Stationary Parameters

Consider the 1one-period interaction uncapacitated problem with stationary parameters (that is, $s_t =$, $d_t^o = d^o$, $\alpha_t = \alpha$, $c_t = c$, $h_t = h$ for $t = 1, \ldots, T$). In other words, the variable production cost, holding cost, proportion of customers that choose to wait for one more period, and new customer demand are the same for all periods. Recall that in Theorem 2, we discussed the optimal pricing plan for this problem. In fact, the optimal pricing plan consists of $T/2$ 2-period optimal pricing sequences when T is even and $(T-3)/2$ 2-period optimal pricing sequences and one 3-period optimal pricing sequence when T is odd. Note that it suffices to consider only a sequence of decreasing prices as a candidate for an optimal pricing sequence. This follows immediately from Lemma 1, in which we argue that regeneration points coincide with periods with price markups. Limiting our attention exclusively to a sequence of decreasing prices, we characterize an n-period optimal pricing sequence that maximizes the profit for n periods in Lemma 2. In Theorem 3, we then show that repeating the 2-period optimal pricing sequence maximizes the average profit; therefore, it is optimal when T is even. All that remains is to deal with the case when T is odd—we address this detail in online Appendix F.

To determine the n-period optimal pricing sequence, let $\mathbf{p}_n$ be a $1 \times n$ vector of decreasing prices (i.e., $p_1 \geqslant p_2 \cdots \geqslant p_n$) and $f_n(\mathbf{p}_n)$ be the firm's profit for n periods under pricing sequence $\mathbf{p}_n$:

$$\begin{aligned} f_n(\mathbf{p}_n) &= f_n(p_1, p_2, \ldots, p_n) \\ &= \sum_{t=1}^{n} (p_t - c)(d^0 - sp_t) \\ &\quad + \alpha \sum_{t=2}^{n} s(p_t - c)(p_{t-1} - p_t). \end{aligned} \tag{5}$$

When $n = 1$, finding the optimal price that maximizes $f_1(p_1) = (p_1 - c)(d^0 - sp_1)$ is trivially given by

$$p^* = \frac{d^0 + sc}{2s}. \tag{6}$$

We call p^* the optimal noninteraction price because there will be no demand interaction if p^* is repeated. If the optimal noninteraction price is repeated for n periods, the firm's profit is simply

$$\begin{aligned} f_n^{NI}(p^*, p^*, \ldots, p^*) &= \sum_{t=1}^{n} (p^* - c)(d^0 - sp^*) \\ &= -ncd^0 + \sum_{t=1}^{n} ((d^0 + sc)p^* - s(p^*)^2) \\ &= n\frac{(d^o - sc)^2}{4s}. \end{aligned} \tag{7}$$

However, in general, the noninteraction price is not optimal, so we derive an expression for an optimal n-period pricing sequence. To do this, first let

$$\begin{aligned} \Delta f_n(\mathbf{p}_n) &= \Delta f_n(p_1, p_2, \ldots, p_n) \\ &= \frac{f_n(p_1, p_2, \ldots, p_n) - f_n^{NI}(p^*, p^*, \ldots, p^*)}{n} \end{aligned}$$

be the average profit increase per period under $\mathbf{p}_n$ over $(p^*, p^*, \ldots, p^*)$. It is clear that a vector of prices that maximizes $f_n(\mathbf{p}_n)$ also maximizes $\Delta f_n(\mathbf{p}_n)$. That is, if $\bar{\mathbf{p}}_n = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$ maximizes $f_n(\mathbf{p}_n)$, then

$$\begin{aligned} \Delta f_n(\bar{\mathbf{p}}_n) &= \frac{f_n(\bar{p}_1, \ldots, \bar{p}_n) - f_n^{NI}(p^*, \ldots, p^*)}{n} \\ &\geqslant \frac{f_n(p_1, \ldots, p_n) - f_n^{NI}(p^*, \ldots, p^*)}{n} = \Delta f_n(\mathbf{p}_n) \end{aligned}$$

for all decreasing prices $\mathbf{p}_n$; therefore, $\bar{\mathbf{p}}_n$ defines an n-period optimal pricing sequence. In online Appendix C, we prove that:

Lemma 2. *An n-period optimal pricing sequence $\bar{\mathbf{p}}_n$ is given by*

$$\bar{p}_i = p^*(1 + \delta_i^*(n)), \quad i = 1 \ldots n, \tag{8}$$

where

$$\begin{aligned} &\delta_0^*(n) = 1 - \frac{c}{p^*} \quad \textit{and} \\ &\delta_i^*(n) = \frac{\beta_{n-i}(n)\delta_{i-1}^*(n) - \delta_0^*(n)}{\beta_{n-i+1}(n)}, \quad i = 1 \ldots n, \end{aligned} \tag{9}$$

and

$$\beta_0(n) = 1, \qquad \beta_1(n) = 2 + \frac{2}{\alpha}, \qquad \textit{and}$$

$$\beta_i(n) = \begin{cases} \left(2 + \dfrac{2}{\alpha}\right)\beta_{i-1}(n) - \beta_{i-2}(n) \\ \quad \textit{for } i = 2 \ldots n-1, \\ \dfrac{2}{\alpha}\beta_{i-1}(n) - \beta_{i-2}(n) \quad \textit{for } i = n. \end{cases} \tag{10}$$

The average profit increase under the n-period optimal pricing sequence (over noninteraction pricing) is

$$\Delta f_n^* = \frac{sp^{*2}\alpha}{n}\left[\frac{a}{2}(\delta_1^*(n) - \delta_n^*(n))\right].$$

We use Δf_n instead of $f_n(\mathbf{p}_n)$ to determine the length of the optimal price sequence that leads to the greatest average profit increase. Note that if an n-period optimal pricing sequence is used, then the increase in the average profit depends only on the difference between the price distortion (from the noninteraction price p^*) in the first period (i.e., $\delta_1^*(n)$) and the price distortion in the last period (i.e., $\delta_n^*(n)$) in the price sequence. We analyze the average profit increases with respect to the length of the price sequence, n, by characterizing the behavior of $\delta_1^*(n)$ and $\delta_n^*(n)$ in n, and then show that the 2-period optimal pricing sequence indeed maximizes the average profit increase. We prove this key result by developing an upper bound on the average profit increase under an n-period optimal pricing sequence for all n. Using several technical lemmas proved in online Appendix D, we characterize the asymptotic properties for both $\delta_1^*(n)$ and $\delta_n^*(n)$, and use these results in online Appendix E to prove the following theorem:

Theorem 3. *The optimal* 2*-period pricing sequence maximizes the average profit increases for the one-period interaction uncapacitated problem with stationary parameters.*

Theorem 3 implies that a periodic pricing strategy that skims off high-valued customers in the first period and serves many of the low-valued customers in the following period is optimal for the infinite horizon one-period interaction problem. Under this high-low pricing strategy, it is optimal to price higher than the noninteraction price (p^*) in the first period with the intention of absorbing surplus from high-valued customers, and then price lower than p^* in the next period to absorb surplus from low-valued customers before some of them exit the market. That is,

$$p^{\text{high}} = p^* + (p^* - c)\left[\frac{\alpha(\alpha+2)}{4\alpha+4-\alpha^2}\right] \geqslant p^*,$$
$$p^{\text{low}} = p^* + (p^* - c)\left[\frac{\alpha(\alpha-2)}{4\alpha+4-\alpha^2}\right] \leqslant p^*, \tag{11}$$

where p^*, c, and α denote, respectively, the optimal noninteraction price, the unit production cost, and the portion of unsatisfied customers transferred from the previous period. Note that both inequalities become strict when $\alpha > 0$.

Although a cyclic pricing pattern is conceptually similar to the results in intertemporal price discrimination models of Sobel (1991) and Conlisk et al. (1984), there are several key differences. In particular, the fact that customers remain in the market for a finite number of periods plays a crucial role in the length of an optimal pricing sequence and prices within the sequence. First, as we discussed previously, the spread of reservation prices between high-valuation and low-valuation customers affects the length of an optimal pricing sequence in their models. In our model, the effect that customers will stay one more period ($K = 1$) results in a 2-period pricing sequence regardless of the spread of reservation price. Second, in their models, customers never leave the market until they purchase the product; thus, the seller always finds it optimal to sell to all customers. But, depending on the proportion of high-type customers in a cohort, the prices in an optimal pricing sequence can be either always above or below the optimal noninteraction price. On the other hand, customers in our model will leave the market after one period, thus, the nonsale price p^{high} is always higher than the optimal noninteraction price to skim off the top and the markdown price p^{low} is always less than the optimal noninteraction price to sell to customers with low reservation price. Even when the seller follows an optimal pricing sequence, not all customers are served.

In a finite horizon problem, repeating this high-low pricing strategy is optimal when T is even. On the other hand, such a strategy would be cut short when T is odd, so a small modification of the pricing strategy is required. Indeed, when T is odd, a strategy that consists of 2-period optimal pricing sequences for $T-3$ periods and a 3-period optimal price sequence for the last three periods is optimal, as we show below. When a 3-period optimal pricing sequence is used, it is interesting to note that prices offered in the first two periods are higher than p^*, while the price in the third period is lower than p^* to sell to customers with low reservation prices. It is easy to verify that

$$p^{3\text{-high}} = p^* + (p^* - c)\left[\frac{\alpha(\alpha^2+4\alpha+2)}{2\alpha^2+8\alpha+4-\alpha^3}\right] \geqslant p^*,$$
$$p^{3\text{-med}} = p^* + (p^* - c)\left[\frac{\alpha^3}{2\alpha^2+8\alpha+4-\alpha^3}\right] \geqslant p^*,$$
$$p^{3\text{-low}} = p^* + (p^* - c)\left[\frac{\alpha(\alpha^2-2\alpha-2)}{2\alpha^2+8\alpha+4-\alpha^3}\right] \leqslant p^*.$$

Finally, in online Appendix F, we complete the proof of Theorem 2 by addressing the odd horizon length case.

4.2. Model 2-($K = 1$, $Q = \infty$) with Time-Varying Parameters

In contrast to the case with stationary parameters, no closed-form solution exists for time-varying parameters. Instead, we develop an efficient algorithm to determine the optimal pricing plan using Lemma 1. Recall that the regeneration point conditions are equivalent to price markups in the one-period interaction price. That is, if the price in any period t is greater than or equal to the price charged in the previous period, period t is a regeneration point and the original problem can be decomposed into two subproblems (one up to period $t-1$ and the other from period t). Every pricing sequence between two regeneration points must be monotonically decreasing and we develop a shortest-path-based algorithm using this observation.

Without loss of generality, we consider a problem satisfying a nonspeculative production condition $c_t \leqslant c_{t-1} + h_{t-1}$, $t = 1, \ldots, T$, because any arbitrary problem instance can be reformulated as an equivalent problem satisfying the condition. With unlimited capacity and the nonspeculative condition, there is no incentive to carry inventory, i.e., $I_t = 0$ for all periods and the production quantity equals demand realized in each period ($x_t = d_t,\ t = 1, \ldots, T$). Then, Model 2 for $K = 1$ can be simplified as follows:

Model 2-($K = 1$, $Q = \infty$) with Time-Varying Parameters

$$\max_{p_t, d_t, r_t^k} \sum_{t=1}^{T} p_t(d_t - c_t) \tag{12}$$

$$\text{s.t.}\quad d_t = r_t^0 + r_t^1,$$

$$0 \leqslant p_t \leqslant d_t^o / s_t,$$

$$r_t^k = \begin{cases} d_t^o - s_t p_t & \text{if } k = 0, \\ \alpha_{t-1} s_{t-1}[p_{t-1} - p_t]^+ & \text{if } k = 1. \end{cases}$$

After substituting for d_t and rewriting the objective function, we get

$$\max_{\substack{0\leqslant p_t \leqslant d_t^o/s_t \\ t=1,\ldots,T}} \left[\sum_{t=1}^{T}(p_t - c_t)(d_t^o - s_t p_t) + \sum_{t=2}^{T}(p_t - c_t)\alpha_{t-1}s_{t-1}[p_{t-1} - p_t]^+\right]. \tag{13}$$

The concavity of (13), together with Lemma 1, enables us to decompose the problem into several independent subproblems, each of which starts at a regeneration point and contains a monotonically decreasing pricing sequence. Let $R = \{t_1 = 1, t_2, \ldots, t_n = T+1\}$ be the set of regeneration periods. Then, Equation (13) can be rewritten as the sum of $n-1$ independent maximization problems:

$$\max_{\substack{0\leqslant p_t \leqslant d_t^o/s_t \\ t=1,\ldots,T}} \sum_{i=1}^{n-1} \sum_{t=t_i}^{t_{i+1}-1} \left[(p_t - c_t)(d_t^o - s_t p_t) + \alpha_{t-1}s_{t-1} \cdot (p_t - c_t)(p_{t-1} - p_t)\right] \tag{14}$$

subject to $t_1 = 1, t_2, \ldots, t_n = T+1$, are regeneration points.

This suggests that finding the optimal pricing plan is equivalent to finding the optimal set of regeneration points and solving the pricing problem for each subsequence independently; and this, in turn, suggests a shortest-path-based algorithm for this problem (see Figure 4).

Algorithm.

Step 1. Calculate $E_{(i,j)}$ for $1 \leqslant i < j \leqslant T+1$ as follows:

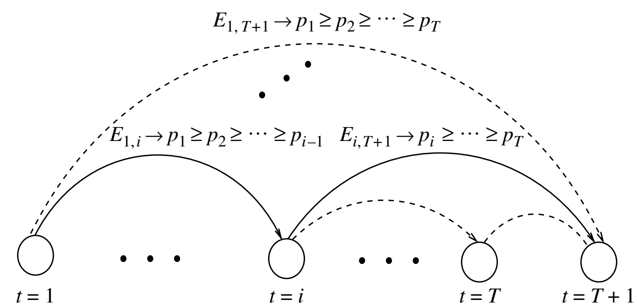
$$E_{(i,j)} = -\max_{\{p_t\}_{t=i}^{j-1}} \left[\sum_{t=i}^{j-1}[(p_t - c_t)(d_t^o - s_t p_t)] + \sum_{t=i+1}^{t-1}[\alpha_{t-1}s_{t-1}(p_t - c_t)(p_{t-1} - p_t)]\right] \tag{15}$$

$$\text{s.t.}\quad p_i \geqslant \cdots \geqslant p_{j-1},$$

$$0 \leqslant p_t \leqslant d_t^o/s_t \quad \text{for all } t = i, \ldots, j-1,$$

and let $(p_t)_i^{j-1} = (p_i, p_{i+1}, \ldots, p_{j-1})$ be the prices determined in $E_{(i,j)}$.

Figure 4. The shortest path that goes through the optimal set of regeneration points.



Step 2. Construct a complete graph G_T with nodes $t = 1, 2, 3, \ldots, T+1$ and weights on arc (i, j) defined as $E_{(i,j)}$.

Step 3. Solve the shortest-path problem on the graph G_T from node 1 to $T+1$.

Let $(t_1 = 1, t_2, t_2, t_3, \ldots, t_{n-1}, t_n = T+1)$ be the shortest path from node 1 to $T+1$. Then, the t_is are regeneration points, the optimal pricing plan is

$$\{p_t^*\}_{t=1}^{T} = \{(p_t)_1^{t_2-1}, (p_t)_{t_2}^{t_3-1}, \ldots, (p_t)_{t_{n-1}}^{T+1}\},$$

and the optimal production plan is

$$\{x_t^*\}_{t=1}^{T} = \{(d_t)_1^{t_2-1}, (d_t)_{t_2}^{t_3-1}, \ldots, (d_t)_{t_{n-1}}^{T+1}\}.$$

Step 1 of the algorithm requires maximization of a concave quadratic objective function with linear inequalities for each (i, j), where $1 \leqslant i < j \leqslant T+1$, which is globally solvable in polynomial time by interior point methods. Let T_C be the time complexity of solving one concave quadratic program. Then, in Step 2 of the algorithm, constructing a complete graph requires $O(T^2 T_C)$. Because the time complexity of solving a shortest-path algorithm on a complete graph is $O(T^2)$, the overall complexity of the algorithm is $O(T^2 T_C)$.

We now show that the algorithm above finds the optimal pricing plan by finding the optimal set of regeneration points. Note that $-E_{(t_i, t_{i+1})}$ is the maximum profit that can be generated from period t_i and period $t_{i+1} - 1$ when any profit from residual demand realized in period t_i is ignored. Let $(p_t)_{t=t_i}^{t_{i+1}}$ and $(x_t)_{t=t_i}^{t_{i+1}}$ be a sequence of decreasing prices and corresponding production quantities which determine $E_{(t_i, t_{i+1})}$. Suppose that two arcs, (t_i, t_{i+1}) and (t_{i+1}, t_{i+2}), are on the shortest path, but t_{i+1} is not a regeneration point, i.e., $p_{t_{i+1}-1} \geqslant p_{t_{i+1}} \geqslant p_{t_{i+1}+1} \geqslant \cdots \geqslant p_{t_{i+2}-1}$. From the definition of $E_{(i,j)}$, $E_{(t_i, t_{i+2})}$ is no greater than $E_{(t_i, t_{i+1})} + E_{(t_{i+1}, t_{i+2})}$, and this contradicts the fact that both arcs are on the shortest path. Therefore, the shortest path will only consist of paths connecting regeneration points, so that the algorithm determines the optimal set of regeneration points as well as the optimal pricing and production plan.

5. Heuristic Pricing Policies for K-Period Interaction

Building on our analysis of the one-period interaction model, we consider the K-period interaction case, in which some portion of demand can remain in the system up to K periods. Although this problem can be decomposed as in the single-period interaction case, a pricing sequence for the K-period problem is not necessarily a monotonically nonincreasing sequence; therefore, regeneration points do not necessarily coincide with price markups. This means that solving for a nondecreasing sequence of prices which maximizes the profit for n periods does not generate an n-period optimal pricing sequence.

Because of this, we focus on an intuitive and efficient pricing scheme based on the result of the one-period interaction model. In particular, we develop heuristic pricing policies consisting of nonincreasing sequences of prices.

5.1. Closed-Form Expression for Unlimited Capacity and Stationary Parameters (Model 2-($Q=\infty$))

As in the one-period interaction case, we start with a K-period interaction model with stationary parameters, and no capacity limitation. In addition, we assume that a constant fraction of unsatisfied customers will wait in the market up to K periods (i.e., $\alpha_k^t = \alpha$ for all $k = 1, \ldots, K$, $t = 1, \ldots, T$). Note that because all parameters are assumed to be stationary, nonanticipation constraints hold for all periods (i.e., $c_{t+1} \leqslant c_t + h_t$, $t = 1, \ldots, T-1$), so that there is no incentive to carry inventory in any period. As a result, demand realized in each period, d_t, must be equal to the production quantity in the same period, x_t. After substituting $x_t = d_t$ into Model 2, we obtain the following objective function for the K-period interaction case:

$$f(\mathbf{p}) = \sum_{t=1}^{T}(p_t - c_t)(d^0 - sp_t) + \sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha s(p_t - c) \cdot [p_{m(t,k)} - p_t] I_{p_{m(t,k)} \geqslant p_t}. \tag{16}$$

In general, a K-period interaction problem cannot be decomposed by decreasing pricing sequences. However, for long enough time horizons, it makes sense that decreasing pricing sequences will perform reasonably well. Consider an n-period sequence of decreasing prices and a pricing plan that repeats this n-period sequence (i.e., an n-period cyclic pricing plan such that $p_{jn+h} = p_t$, $h = 1, \ldots, n-1$, and $j \geqslant 0$). When T is sufficiently large and a multiple of n, no residual demand will be realized between two periods in different cycles. In this case, the only demand interactions realized are the ones caused by decreasing prices within an n-period sequence of decreasing prices. Thus, we are motivated to find a nonincreasing sequence of prices which maximizes the average profit contribution under the assumption that the same sequence will be repeated in a pricing plan. Therefore, the profit from an n-period sequence of decreasing prices can be expressed as

$$f_n(\mathbf{p}) = \sum_{t=1}^{n}(p_t - c)(d^0 - sp_t) + \sum_{t=2}^{n}\min(t-1, K)\alpha s(p_t - c)(p_{t-1} - p_t). \tag{17}$$

Following our single-period interaction result, we give a closed-form expression for the optimal nonincreasing sequences of length n which maximizes Equation (17).

Lemma 3. *Among all sequences of decreasing prices of length n, the following sequence maximizes Equation* (17) *in the K-period interaction case*:

$$p_i = p^*(1+\delta_i), \quad i = 1\ldots n, \tag{18}$$

where δ_i, $i = 1, \ldots, n$, satisfies the following recursive relationship. Define $a = (1 - c/p^*)$. For $n \leqslant K+1$,

$$\delta_i = \begin{cases} 1, & i = 0, \\ (i-1)\dfrac{\beta_{n-i}}{\beta_{n-i+1}}\delta_{i-1} - a * \dfrac{(n-1)!}{(i-1)!}\dfrac{n-1}{\beta_{n-i+1}} + a * \dfrac{\sum_{j=i}^{n-1}((j-1)!/(i-1)!)\beta_{n-j}}{\beta_{n-i+1}}, & i = 1\ldots n-1, \\ \dfrac{(n-1)\beta_0}{\beta_1}\delta_{i-1} - \dfrac{a}{\beta_1}(n-1), & i = n, \end{cases} \tag{19}$$

$$\beta_i = \begin{cases} 1, & i = 0, \\ \beta_1 = 2(n-1+\alpha^{-1}), & i = 1, \\ 2(n-i+\alpha^{-1})\beta_{i-1} - (n-i+1)^2\beta_{i-2}, & i = 2\ldots n. \end{cases} \tag{20}$$

For $n > K+1$,

$$\delta_i = \begin{cases} 1, & i = 0, \\ (i-1)\dfrac{\beta_{n-i}}{\beta_{n-i+1}}\delta_{i-1} - a * \dfrac{K!}{(i-1)!}\dfrac{K^{n-K}}{\beta_{n-i+1}} + a * \dfrac{\sum_{j=i}^{K}((j-1)!/(i-1)!)\beta_{n-j}}{\beta_{n-i+1}}, & i = 1\ldots K, \\ \dfrac{K\beta_{n-i}}{\beta_{n-i+1}}\delta_{i-1} - \dfrac{a}{\beta_{n-i+1}}K^{n-i+1}, & i = K+1\ldots n, \end{cases} \tag{21}$$

$$\beta_i = \begin{cases} 1, & i = 0, \\ 2(K+\alpha^{-1}), & i = 1, \\ 2(K+\alpha^{-1})\beta_{i-1} - (K)^2\beta_{i-2}, & i = 2\ldots n-K, \\ 2(n-i+\alpha^{-1})\beta_{i-1} - (n-i+1)^2\beta_{i-2}, & i = n-K+1, \ldots, n. \end{cases} \tag{22}$$

As in the one-period interaction case, all β_is and δ_is are functions of sequence length n (i.e., $\beta_i(n)$ and $\delta_i(n)$), but we suppress this dependence for ease of exposition. The proof is in online Appendix G.

5.2. Capacity Constraints and Time-Varying Parameters (Model 2)

To find a heuristic solution for the most general Model 2 based on the results and intuition developed in prior sections, we focus on nonincreasing pricing sequences, extend the optimal algorithm from §5.1 to the K-interaction period problem, and then develop a linear program to deal with capacity constraints.

Heuristic Algorithm.

Step 1. Let $\mathrm{P}^h = \{P_t^h\}_{t=1}^T$ be the pricing plan found by solving the following shortest-path problem:

(a) Calculate $E_{(i,j)}$ for $1 \leqslant i < j \leqslant T+1$ as follows:

$$E_{(i,j)} = -\max_{\{x_t, I_t, p_t\}_{t=i}^{j-1}} \left[\sum_{t=i}^{j-1} [p_t(d_t^o - s_t p_t)] + \sum_{t=i+1}^{j-1} \sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k} \cdot [p_t(p_{t-1} - p_t)] - \sum_{t=i}^{j-1} c_t x_t - \sum_{t=i}^{j-1} h_t I_t \right]$$

$$\begin{aligned} \text{s.t.}\quad & I_{t-1} + x_t = I_t + [d_t^o - s_t p_t] + \sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}(p_{t-1} - p_t) \quad \forall\, t = i, \ldots, j-1, \\ & p_t \leqslant d_t^o / s_t \quad \forall\, t = i, \ldots, j-1, \\ & p_t \geqslant p_{t+1} \quad \forall\, t = i, \ldots, j-2, \\ & x_t \leqslant Q_t \quad \forall\, t = i, \ldots, j-1, \\ & x_t, I_t, p_t \geqslant 0 \quad \forall\, t = i, \ldots, j-1, \\ & I_{i-1} = 0, \end{aligned}$$

and let $(P_t^h)_i^j = (p_i^h, p_{i+1}^h, \ldots, p_{j-1}^h)$ be the prices determined in $E_{(t_1,t_2)}$.

(b) Construct a complete graph G_T with nodes $t = 1, 2, 3, \ldots, T+1$ and weights on arc (t_1, t_2) defined as $E_{(t_1,t_2)}$.

(c) Solve the shortest-path problem on the graph G_T from node $t = 1$ to $t = T+1$. Let $(t_1 = 1, t_2, t_3, \ldots, t_{n-1}, t_n = T+1)$ be the shortest path from node $t_1 = 1$ to $t_n = T+1$. Then, construct a pricing plan

$$\{P_t^h\}_{t=1}^T = \{(P_t^h)_1^{t_2}, (P_t^h)_{t_2}^{t_3}, \ldots, (P_t^h)_{t_{n-1}}^{T+1}\}.$$

Step 2. Calculate the demand, D_t^h, for each period generated by the heuristic pricing sequence as follows:

$$D_t^h = \begin{cases} [d_t^o - s_t P_t^h] & \text{if } t = 1, \\ [d_t^o - s_t P_t^h] + \displaystyle\sum_{k=1}^{\min\{t-1,K\}} \alpha_k^{t-k} s_{t-k} \cdot \Big[\min_{i \in \{1,\ldots,k\}} P_{t-i}^h - P_t^h \Big]^+ & \text{if } 2 \leqslant t \leqslant T. \end{cases} \tag{23}$$

Step 3. Solve the following linear program to determine optimal allocation of capacity:

$$\max_{\{x_t, d_t, I_t\}_{t=1}^T} \quad \sum_{t=1}^T P_t^h d_t - \sum_{t=1}^T c_t x_t - \sum_{t=1}^T h_t I_t \tag{24a}$$

$$\text{s.t.}\quad x_t + I_{t-1} = d_t + I_t, \tag{24b}$$

$$\begin{aligned} & x_t \leqslant Q_t, \\ & d_t \leqslant D_t^h, \\ & x_t, d_t, I_t \geqslant 0 \quad \forall\, t = 1, \ldots, T. \end{aligned}$$

In Step 1(a) of the algorithm, we define a concave quadratic function, $E_{(i,j)}$, to approximate the profit accrued between i and $j-1$. The first term in $E_{(i,j)}$ is the revenue from the demand originated and realized in the same period, the second term represents the revenue from the residual demand, and the third and fourth terms are production and inventory costs. We find $(p_i, \ldots, p_{j-1})$ which maximizes $E_{(i,j)}$ under the following constraints: initial inventory and inventory balance constraints, nonnegative demand constraints ($p_t \leqslant d_t^o/s_t$), fixed-ordering constraints, and capacity constraints. We also assume that all variables are nonnegative.

Step 1(a) of the algorithm requires maximization of a concave quadratic objective function with linear inequalities for each (i, j), where $1 \leqslant i < j \leqslant T+1$, which is globally solvable in polynomial time. Let T_C be the time complexity of solving one concave quadratic program. Then, in Step 1(b) of the algorithm, constructing a complete graph requires $O(T^2 T_C)$. Because time complexity of solving a shortest-path algorithm on a complete graph is $O(T^2)$, the overall complexity of Step 1 of the heuristic algorithm is $O(T^2 T_C)$.

Note that for the problem with time-varying parameters, nonanticipative costs, and no capacity constraints, this algorithm without Step 3 finds the best solution among solutions with nonincreasing pricing sequences. Step 3 transforms this solution into a feasible solution for the capacity-constrained problem. In §6, we test the performance of this heuristic.

6. Computational Study

We conducted a computational study to develop insights into the impact of demand interaction on profit and manufacturing decisions, as well as the effectiveness of our heuristics. We created 972 problem instances by varying problem parameters and demand assumptions and tested three different policies: the optimal pricing/manufacturing policy that explicitly takes demand interactions into account, our proposed heuristic policy, and a myopic policy that ignores demand interactions (that is, assuming $K = 0$) and follows a pricing schedule determined by a myopic algorithm.

6.1. Design of the Computational Study

We first consider a 6-period planning horizon (T = 6) and created a set of experiments by varying several model parameters and demand patterns.

Demand Curve Scenarios. We examine four scenarios for demand variations: stationary, increasing, decreasing, and seasonal demand. We consider a standard demand curve, $D_t(p_t) = d_t^0 - s_t p_t$, where d_t^0 and s_t are, respectively, demand at zero price and sensitivity of demand to price variations. Depending on the demand scenario, we vary d_t^0 in different ways, while keeping p_t^0 constant. This corresponds to a series of linear demand curves whose intersection with the demand axis is changing in three different fashions (seasonal, increasing, and decreasing) around the standard demand curve, while intersection with price axis is constant throughout the periods. For computational study, we use $D_q(p_t) = 30 - p_t$ as our standard demand price curve. The actual demand curves for each scenario follow (see Figure (5)):

(1) Stationary demand curve scenario: $D_1(p_1) = D_2(p_2) = D_3(p_3) = D_4(p_4) = D_5(p_5) = D_6(p_6) = 30 - p$.

(2) Increasing demand curve scenario: $D_1(p_1) = 15 - \frac{5}{10}p_1$; $D_2(p_2) = 21 - \frac{7}{10}p_2$; $D_3(p_3) = 27 - \frac{9}{10}p_3$; $D_4(p_4) = 33 - \frac{11}{10}p_4$; $D_5(p_5) = 39 - \frac{13}{10}p_5$; $D_6(p_6) = 45 - \frac{15}{10}p_6$.

(3) Decreasing demand curve scenario: $D_1(p_1) = 45 - \frac{15}{10}p_1$; $D_2(p_2) = 39 - \frac{13}{10}p_2$; $D_3(p_3) = 33 - \frac{11}{10}p_3$; $D_4(p_4) = 27 - \frac{9}{10}p_4$; $D_5(p_5) = 21 - \frac{7}{10}p_5$;$D_6(p_6) = 15 - \frac{5}{10}p_6$.

(4) Seasonal demand curve scenario: $D_1(p_1) = 15 - \frac{5}{10}p_1$; $D_2(p_2) = 30 - p_2$; $D_3(p_3) = 45 - \frac{15}{10}p_3$; $D_4(p_4) = 45 - \frac{15}{10}p_4$; $D_5(p_5) = 30 - p_5$; $D_6(p_6) = 15 - \frac{5}{10}p_6$.

Interaction Levels. $(K = 1, 2, 3)$.

Capacity Levels. We keep capacity constant over the planning horizon, although at different levels. Those levels are defined loosely as uncapacitated, mildly capacitated, and strictly capacitated. Because we use $D_q(p_t) = 30 - p_t$ as the base demand curve, the optimal noninteraction price when $c = 0$ is $p^* = 15$ and the demand at $p^* = 15$ is $D_q(p^*) = 15$. We set capacity levels first at infinity (uncapacitated case), then at 15 (mildly capacitated case), and finally at 5 (strictly capacitated case). For the demand curves given above, these values are calculated as follows:

(1) No capacity constraint: for this range of parameters, no capacity is equivalent to the case where capacity is 100.

(2) Medium capacity constraint: capacity = 15.

(3) Tight capacity constraint: capacity = 5.

Production Costs. Low production cost ($c = 0$), medium production cost ($c = 5$), and high production cost ($c = 10$). We assume that production cost is constant over the planning horizon.

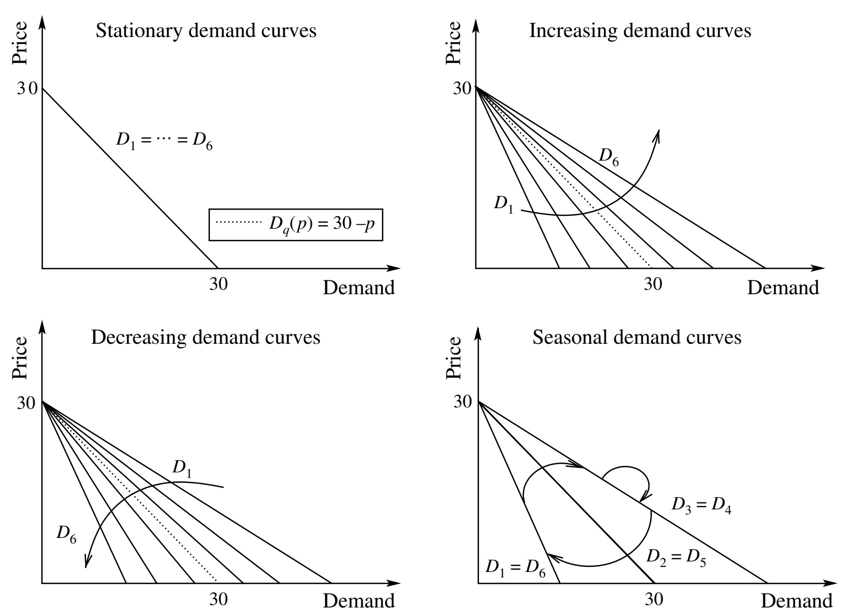
Inventory Holding Costs. Low inventory holding cost ($h = 1$), medium inventory holding cost ($h = 2$), and high inventory holding cost ($h = 10$). We assume that the holding cost is constant over the planning horizon.

Customer Waiting Proportions. $\alpha_k^t = \alpha^k$ for all $t = 1, \ldots, T$ and $k = 1, \ldots, K$, where $\alpha = 1$ (high), 0.5 (medium), and 0.2 (low).

6.2. Analysis

For our analysis, we solve each problem instance using three different policies: the optimal policy, our heuristic, and a myopic policy. Let Π^{OPT}, Π^H, and Π^M be profits generated by the optimal, heuristic, and myopic algorithms, respectively. To compare how the optimal policy (or the heuristic policy) fares with the myopic policy, we use the percentage improvement in profit over myopic pricing

Figure 5. The price-demand curves for each scenario.



as follows. Note that even when the policy is determined myopically, the profit is determined assuming that there is demand interaction in the underlying system:

$$\Delta_{\mathrm{OPT}} = \frac{\Pi^{\mathrm{OPT}} - \Pi^M}{\Pi^M} * 100,$$

$$\Delta_{\mathrm{HEU}} = \frac{\Pi^{H} - \Pi^M}{\Pi^M} * 100.$$

Note that $\Delta_{\mathrm{OPT}} - \Delta_{\mathrm{HEU}}$ represents a gap between an optimal policy and a heuristic.

We found an optimal policy using Theorem 1 and the algorithm described in Corollary 1. This algorithm enumerates all possible orderings of price sequences and solves a concave maximization problem for each ordering. Because an optimization problem with a quadratic and concave objective function and a set of linear constraints is polynomially solvable with complexity of $O(T_C)$, the complexity of the optimal algorithm is $O(T!T_C)$. On the other hand, the complexity of our heuristic algorithm based on the shortest-path computation is $O(T^2 T_C)$, and the complexity of a myopic algorithm is simply $O(T_C)$.

Under the myopic algorithm, we first solve the pricing model as if no demand interaction exists, then determine the demand realization (assuming demand interaction) generated by the myopic pricing solution. Finally, we solve an LP to determine the final production decisions.

Myopic Algorithm.

Step 1. Let $\mathbf{p^m} = \{\mathbf{P^m_t}\}_{\mathbf{t=1}}^{\mathbf{T}}$ be the pricing plan found by solving the following concave maximization problem:

$$\begin{aligned} M_T = \max_{\{p_t, x_t, I_t\}_{t=1}^T} & \left[\sum_{t=1}^{T}[p_t(d_t^o - s_t p_t)] - \sum_{t=1}^{T} c_t x_t - \sum_{t=1}^{T} h_t I_t\right] \\ \text{s.t.}\quad & I_{t-1} + x_t = I_t + [d_t^o - s_t p_t] \quad \forall\, t = 1, \ldots, T, \\ & p_t \leqslant d_t^o / s_t \quad \forall\, t = 1, \ldots, T, \\ & x_t \leqslant Q_t \quad \forall\, t = 1, \ldots, T, \\ & x_t, I_t, p_t \geqslant 0 \quad \forall\, t = 1, \ldots, T. \end{aligned}$$

Step 2. Calculate the demand, D_t^m, for each period generated by the myopic pricing sequence as follows:

$$D_t^m = \begin{cases} [d_t^o - s_t P_t^m] & \text{if } t = 1, \\ [d_t^o - s_t P_t^m] + \displaystyle\sum_{k=1}^{\min\{t-1, K\}} \alpha_k^{t-k} s_{t-k} & \\ \quad \cdot \left[\displaystyle\min_{i \in \{1,\ldots,k\}} P_{t-i}^m - P_t^m\right]^+ & \text{if } 2 \leqslant t \leqslant T. \end{cases}$$

Step 3. Solve the following linear program to determine optimal allocation of capacity:

$$\begin{aligned} \Pi^M = \max_{\{x_t, d_t, I_t\}_{t=1}^T} & \sum_{t=1}^{T} P_t^m d_t - \sum_{t=1}^{T} c_t x_t - \sum_{t=1}^{T} h_t I_t \\ \text{s.t.}\quad & x_t + I_{t-1} = d_t + I_t, \\ & x_t \leqslant Q_t, \\ & d_t \leqslant D_t^m, \\ & x_t, d_t, I_t \geqslant 0 \quad \forall\, t = 1, \ldots, T. \end{aligned}$$

We used the CPLEX(R) solver running on an Intel(R) Pentium(R) 4 Mobile CPU 1.60 GHz computer with 512 MB RAM and 30 GB HDD to conduct the computational study.

6.2.1. The Impact of Parameters on Optimal Policy. We present the results of our numerical study in Tables 1–4. We first examine how each problem parameter affects the benefit of accounting for demand interaction when making pricing and production plans. To this end, we compute Δ_{OPT} for various scenarios and observed the following.

• On average, the gain from using the optimal policy (Δ_{OPT}) is quite significant. The average gain of the optimal policy is 6.73%. There are many instances where the gain is 20% or more. On average, the gain from the optimal policy (Δ_{OPT}) decreases as the capacity becomes more constrained. This is intuitive because when capacity is tight, the firm can only sell to customers with high reservation prices anyway. When the firm has excess capacity, on the other hand, the

Table 1. Δ_{OPT} and Δ_{HEU} w.r.t. various capacity and interaction levels (i.e., K-values).

		Δ_{OPT} (%)				Δ_{HEU} (%)			
Capacity	Interaction level	Mean	Min	Max	SD	Mean	Min	Max	SD
UnCap	1-period	6.53	0.71	15.95	5.77	6.53	0.71	15.95	5.77
	2-period	10.99	1.19	26.96	9.69	10.99	1.19	26.96	9.69
	3-period	13.73	1.40	35.56	12.26	13.62	1.40	35.56	12.31
MedCap	1-period	4.65	0.15	14.36	4.21	4.57	−0.21	14.36	4.21
	2-period	7.69	0.15	23.08	6.79	7.59	−0.05	23.08	6.78
	3-period	9.48	0.15	31.53	8.47	9.34	−0.02	31.53	8.45
TightCap	1-period	1.83	0.00	6.79	1.89	1.71	−0.69	6.79	1.92
	2-period	2.65	0.00	10.05	2.68	2.57	−0.69	10.05	2.73
	3-period	3.04	0.00	11.52	3.06	2.94	−0.69	11.52	3.09

Note. UnCap/MedCap/TightCap: no capacity/medium capacity/tight capacity.

Table 2. Δ_{OPT} and Δ_{HEU} w.r.t. various capacity and the proportion of customers who stay in the market (i.e., α values).

		Δ_{OPT} (%)				Δ_{HEU} (%)			
Capacity	α	Mean	Min	Max	SD	Mean	Min	Max	SD
UnCap	High	22.77	12.68	35.56	7.23	22.77	12.68	35.56	7.23
	Medium	6.99	3.77	11.43	2.32	6.90	3.77	11.43	2.25
	Low	1.50	0.71	2.81	0.61	1.48	0.71	2.81	0.60
MedCap	High	14.97	2.97	31.53	6.39	14.81	2.97	31.53	6.41
	Medium	5.47	0.93	12.00	2.83	5.36	0.92	12.00	2.84
	Low	1.40	0.15	5.71	1.00	1.33	−0.21	5.71	1.00
TightCap	High	3.80	0.00	11.52	3.09	3.73	0.00	11.52	3.11
	Medium	2.38	0.00	8.16	2.44	2.28	−0.63	8.16	2.49
	Low	1.33	0.00	4.90	1.53	1.22	−0.69	4.90	1.55

Table 3. Δ_{OPT} and Δ_{HEU} w.r.t. various capacity and inventory holding-cost parameters.

		Δ_{OPT} (%)				Δ_{HEU} (%)			
Capacity	Inv. holding cost	Mean	Min	Max	SD	Mean	Min	Max	SD
UnCap	Low	10.42	0.71	35.56	10.07	10.38	0.71	35.56	10.08
	Medium	10.42	0.71	35.56	10.07	10.38	0.71	35.56	10.08
	High	10.42	0.71	35.56	10.07	10.38	0.71	35.56	10.08
MedCap	Low	8.04	0.35	31.53	7.49	7.83	−0.18	31.53	7.48
	Medium	7.23	0.15	31.21	6.86	7.14	−0.21	31.21	6.84
	High	6.55	0.19	31.21	6.58	6.53	0.19	31.21	6.57
TightCap	Low	2.77	0.00	11.52	2.56	2.58	−0.69	11.52	2.67
	Medium	2.46	0.00	10.97	2.58	2.35	−0.40	10.97	2.60
	High	2.28	0.00	10.87	2.75	2.28	0.00	10.87	2.75

Table 4. Δ_{OPT} and Δ_{HEU} w.r.t. various capacity and production cost parameters.

		Δ_{OPT} (%)				Δ_{HEU} (%)			
Capacity	Production cost	Mean	Min	Max	SD	Mean	Min	Max	SD
UnCap	Low	10.42	0.71	35.56	10.07	10.38	0.71	35.56	10.08
	Medium	10.42	0.71	35.56	10.07	10.38	0.71	35.56	10.08
	High	10.42	0.71	35.56	10.07	10.38	0.71	35.56	10.08
MedCap	Low	6.46	0.15	24.85	6.04	6.28	−0.21	24.85	6.04
	Medium	6.67	0.35	26.79	6.32	6.58	0.23	26.79	6.27
	High	8.70	0.52	31.53	8.26	8.64	0.52	31.53	8.23
TightCap	Low	1.90	0.00	6.77	1.91	1.82	−0.41	6.77	1.94
	Medium	2.39	0.00	8.53	2.40	2.30	−0.52	8.53	2.44
	High	3.23	0.00	11.52	3.25	3.10	−0.69	11.52	3.30

Table 5. Relative frequency table of $\Delta_{\mathrm{OPT}} - \Delta_{\mathrm{HEU}}$.

	[0–0.05%][1]	(0.05–0.10%]	(0.10–0.20%]	(0.20–0.40%]	(0.40–0.80%]	(0.80–1.60%]	(1.60–3.20%]
N^2	758	31	50	61	60	10	2
F^3 (%)	77.98	3.19	5.14	6.28	6.17	1.03	0.21

[1]Class Intervals for $\Delta_{\mathrm{OPT}} - \Delta_{\mathrm{HEU}}$.
[2]Number of cases where $\Delta_{\mathrm{OPT}} - \Delta_{\mathrm{HEU}} \in$ class interval.
[3]Percentage of cases where $\Delta_{\mathrm{OPT}} - \Delta_{\mathrm{HEU}} \in$ class interval.

Table 6. Δ_{HEU} w.r.t. various demand scenarios, planning horizons, and capacity values.

		Planning horizon							
		$T=6$ Descriptive statistics of Δ_{HEU}				$T=12$ Descriptive statistics of Δ_{HEU}			
Demand curve scenarios	Capacity	Mean	Min	Max	SD	Mean	Min	Max	SD
Stationary	UnCap	9.76	0.84	27.09	9.28	10.14	0.84	30.14	9.81
	MedCap	6.85	0.35	22.95	6.42	6.96	0.35	24.85	6.54
	TightCap	0.49	0.00	3.37	0.82	0.48	0.00	3.33	0.81
Increasing	UnCap	8.48	0.71	23.42	8.05	9.08	0.74	26.71	8.75
	MedCap	4.24	−0.21	15.91	4.48	4.69	−0.17	19.38	4.86
	TightCap	0.15	−0.69	2.69	0.70	0.32	−0.56	3.10	0.79
Decreasing	UnCap	11.66	0.99	33.66	11.09	11.41	0.95	33.83	10.93
	MedCap	9.15	0.99	31.53	8.34	8.38	0.94	27.17	7.27
	TightCap	4.83	1.07	11.52	2.54	3.88	0.76	10.00	2.24
Seasonal	UnCap	11.62	0.90	35.56	11.24	10.67	0.87	31.81	10.27
	MedCap	8.43	0.35	28.11	7.15	7.50	0.03	27.07	6.95
	TightCap	4.15	1.10	8.34	1.81	3.39	0.65	8.15	1.74

optimal policy improves the profit by selling to residual demand.

- We also notice that Δ_{OPT} increases as the demand interaction level (represented by K) increases (Table 1). As the demand interaction level increases, the size of residual demand also increases, thus the firm's profit from selling to residual demand also increases. Similarly, Δ_{OPT} increases as the proportion of customers who stay in the market increases (Table 2).
- As expected, Δ_{OPT} decreases as holding or production cost increases. Increasing operating cost decreases the range within which intertemporal price discrimination can take place, thus decreasing the benefit of dynamic pricing (Tables 3 and 4).

6.2.2. Heuristic Performance. As demonstrated in Tables 1–4, our heuristic policy performs very well. In many cases, our heuristic either coincides with the optimal policy or is very close to the optimal policy. As illustrated in Table 5, in 98.76% of the instances we ran, the gap is within 0.8%. Out of 972 problem instances, the worst case is 3.2%.

Furthermore, the insights we developed by comparing the optimal policy with the myopic policy continue to hold for the heuristic. Δ_{HEU} decreases as capacity gets tight (Tables 1–4), increases as demand interaction level (K) increases (Table 1), increases as α increases (Table 2), and decreases as holding cost or production cost increases (Tables 3 and 4).

In a relatively small number of cases, the heuristic algorithm diverges from optimality. These cases are characterized by:

- Tight capacity: the decomposition ignores inventory carryover between subproblems, and this may lead to heuristic errors when capacity is tight.
- High K: when K is strictly greater than one, there might be residual demand between two subproblems because nonincreasing sequences do not decompose the problem into subproblems.

Also, for a small number of cases (41 of 972 cases), the performance of the heuristic policy is either equal to or slightly less than that of the myopic algorithm, i.e., $\Delta_{\mathrm{HEU}} \leqslant 0$. As shown in Table 6, the negative values appear only when the capacity is tight and the firm faces increasing demand patterns. Recall that the heuristic algorithm solves a shortest-path problem, and we find the best nonincreasing pricing sequence to represent the profit of each arc. However, the increasing demand curve scenario under tight capacity limitations leads to an optimal pricing policy that is increasing in time with no residual demand. Hence, the myopic policy generates more profit by considering only current demand and solving the entire problem without decomposition.

Finally, to evaluate the performance of the heuristic algorithm for different planning horizon lengths, we conducted additional computational experiments with the same set of parameters, except that the length of the horizon $T = 12$. Because the complexity of computing the optimal policy grows exponentially with T, for this case we only compared heuristic and myopic policies.

As Table 6 illustrates, the relative performance of the heuristic algorithm over the myopic algorithm remains significant as the planning horizon increases from $T = 6$ to $T = 12$.

7. Extensions, Conclusions, and Further Research

7.1. Pricing-Pattern Aware Customers

As we mentioned in the introduction, one reasonable critique of this model is that it assumes that consumers are not aware of impending price decreases. Thus, this model is intended to capture situations in which customers place a high value on a good's availability, and tend to buy it as soon as their budget constraint (i.e., reservation price) is met. We can alternatively model a situation in which customers are aware of the pricing pattern. In this case, they enter the system and stay for at most K periods, and actually make a purchase when their discounted utility is maximized. We model utility as the difference between modified price and reservation price, where $\beta_j \geqslant 1$ is a factor that we multiply by price in the future to represent the disutility of waiting j periods. In other words, all of the customers who arrive at time t and eventually make a purchase will make their purchase at time t^*, where

$$t^* = \arg\min_{t \leqslant l \leqslant t+K} \beta_{(l-t)} p_l.$$

Thus, the revenue from demand in period t is

$$p_{t^*}(d_t^0 - s_t \beta_{(t^*-t)} p_{t^*}).$$

We let $\tilde{d}_i^j$ represent the demand from customers that arrive in period i if they make their purchase j periods in the future, and let y_t^j be a binary variable that takes on the value of one if the utility from making a purchase j periods in the future is the maximum utility over periods $t, t+1, t+2, \ldots, \min(t+K, T)$. Then, this model can be expressed as described below. Once again, we consider a discrete-time multiperiod production system where at each period we decide both price of the product, p_t, and the production quantity of the product, x_t. As before, our objective is to maximize the net profit (25a) subject to inventory balance (25b), production capacity (25c), and demand realization constraints (25e)–(25i):

$$\max_{p_t, x_t, I_t, d_t, \tilde{d}_t^j} \sum_{t=1}^{T} \sum_{j=0}^{\min(K,T-t)} p_{t+j} y_t^j \tilde{d}_t^j - \sum_{t=1}^{T} c_t x_t - \sum_{t=1}^{T} h_t I_t \qquad (25a)$$

$$\text{s.t.} \quad x_t + I_{t-1} = d_t + I_t, \quad t = 1, 2, \ldots, T,\ t_o = 0, \qquad (25b)$$

$$x_t \leqslant Q_t, \quad t = 1, 2, \ldots, T, \qquad (25c)$$

$$d_t = \sum_{j=0}^{\min(K,T-t)} \tilde{d}_{t-j}^j, \quad t = 1, 2, \ldots, T, \qquad (25d)$$

$$\tilde{d}_t^j = d_t^0 - s_t \beta_j p_{t+j} + (1 - y_t^j) \cdot M \quad t = 1, 2, \ldots, T, \quad j = 0, 1, 2, \ldots, \min(K, T-t), \qquad (25e)$$

$$p_t \leqslant d_t^0 / s_t, \quad t = 1, 2, \ldots, T, \qquad (25f)$$

$$\tilde{d}_t^j \geqslant y_t^j \cdot d_t^0, \quad t = 1, 2, \ldots, T, \quad j = 0, 1, 2, \ldots, \min(K, T-t), \qquad (25g)$$

$$\sum_{j=1}^{\min(K,T-t)} y_t^j = 1, \quad t = 1, 2, \ldots, T, \qquad (25h)$$

$$y_t^j \in \{0, 1\}, \quad t = 1, 2, \ldots, T, \quad j = 0, 1, 2, \ldots, \min(K, T-t), \qquad (25i)$$

$$x_t, I_t, d_t, p_t \geqslant 0, \quad t = 1, 2, \ldots, T, \qquad (25j)$$

$$\tilde{d}_t^j \geqslant 0, \quad t = 1, 2, \ldots, T, \quad j = 0, 1, 2, \ldots, \min(K, T-t). \qquad (25k)$$

Note that constraints (25e)–(25i) ensure that y_j^i takes the appropriate value and that M is a large constant.

Gümüş (2007) develops properties of this model, and an optimal (exponential) algorithm that is similar to the algorithms we have developed for our original model in this paper. However, we observe here that the uncapacitated version of this model with stationary parameters is much simpler to analyze than the equivalent formulation of the original model. Because all newly arriving demand that is met before customers exit the system is met in the same period, a simple contradiction-based argument can be used to prove the following result:

Theorem 4. *For the pricing-aware customer model with unlimited capacity and stationary parameters, each period's price will be identical and equal to the noninteraction price,*

$$p^* = \frac{d^0 + sc}{2s}. \qquad (26)$$

Interestingly, although Theorem 4 applies in a more general setting than Theorem 2, Theorems 2 and 4 can conceptually be interpreted as supporting the use of two very different retail pricing strategies. When customers place a high value on a good's availability and tend to buy it as soon as their budget constraint is met, relatively frequent sales can serve to maximize profits, as indicated by Theorem 2. This strategy tends to be employed, for example, by high-end department stores, whose customers may place a high value on the good's availability. Discount stores such as Wal-Mart, however, may employ everyday low-price strategies because their customers are more adequately modeled by the model introduced in this section—they are willing to wait for the lowest possible price.

7.2. Conclusions and Future Research

In this paper, we have demonstrated that it can be valuable to model demand interactions in production/pricing models, and we presented several possible ways to model this interaction. We have presented structure, algorithms, and heuristics for our models. Our computational analysis helps to characterize the value of accounting for demand interaction when making pricing and production decisions in various cost and capacity settings.

Our models have a variety of limitations. These are deterministic models, although actual problems will typically be stochastic. In addition, while we assume that all customers in the market follow the pattern of the behavior described either in our main model or in the model introduced in the previous subsection, we believe that most actual markets have a mixture of customers following the behavior characterized in both models. Nevertheless, we think that the models and results presented in this paper are a good initial step toward capturing the impact of demand interaction in production/pricing models.

Building on our initial framework, we are currently extending these models in several ways. We are working to extend more of our 1-period results to the general K-period case. We are also considering models with fixed setup costs associated with production orders. Our framework allows us to easily integrate setup costs into the model. We are also working to extend our models to a multiproduct setting where a manufacturer produces more than one product and demand for each product is a function of its own price and the prices of other products.

While our model has focused on the problem of a monopolist, a game-theoretic model where multiple manufacturers compete for the same potential demand is worth investigating. This extension will help us to understand the effect of dynamic pricing as a tool to achieve competitive advantage. Finally, we hope to extend our models to the case of stochastic demand.

8. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

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Online Appendix

A. Proof of Theorem 1

We first note that a set of *fixed ordering constraints*, Γ, completely defines which binary variables, y_t^k, take non-zero values. Furthermore, this ordering also defines whether the price p_t is the minimum price quoted for the last k periods from period t. All pricing plans that are feasible under these fixed ordering constraints have the same periods' residual demands realized, and demand realization constraints- (2.3a) to (2.3e) can be rewritten as a set of linear equalities.

In order to show that the objective function is concave for all feasible prices in Model (2)-Γ, it is enough to consider only revenue terms, $\sum_{t=1}^{T} p_t d_t$, since the production and holding costs are linear and concave. By decomposing the revenue into two parts – the revenue generated from the current period's demand and the revenue generated from residual demand, we can rewrite the revenue function under a pricing plan $\mathrm{P} = (p_1, p_2, \ldots, p_T)$ as follows:

$$f(\mathrm{P}) = f(p_1, p_2, \ldots, p_T) = \sum_{t=1}^{T} p_t d_t = \sum_{t=1}^{T} p_t r_t^0 + \sum_{t=1}^{T} \sum_{k=1}^{\min(t-1,K)} p_t r_t^k. \tag{EC.A1}$$

Notice that there are two sources of demand for each period. The first is demand originating from the current period, $r_t^0 = d_t^o - s_t p_t$, and the second is residual demand originating from some period k-periods earlier, $(1 \leqslant k \leqslant K)$, for the current period, $r_t^k = \alpha_k^{t-k} s_{t-k}[\min_{i=1,\ldots,k}(p_{t-i}) - p_t]$ if $\min_{i=1,\ldots,k}(p_{t-i}) \geqslant p_t$. Note that a set of fixed ordering constraints Γ completely determines $\arg\min_{i=1,\ldots,k}(p_{t-i})$ for all pricing plans satisfying the same constraints, PS_Γ. In other words, for any two pricing plans P and $\bar{\mathrm{P}}$ in PS_Γ, $\arg\min_{i=1,\ldots,k}(p_{t-i}) = \arg\min_{i=1,\ldots,k}(\bar{p}_{t-i})$ for all t and k. Let $m(t,k)$ be the index of the minimum price, that is, $\arg\min_{i=1,\ldots,k}(p_{t-i})$ for all $k \geqslant 1$. Then, residual demand becomes $r_t^k = \alpha_k s_{t-k}[p_{m(t,k)} - p_t]$. Substituting r_t^k into the revenue function, we get the following;

$$f(\mathrm{P}) = \sum_{t=1}^{T} p_t(d_t^0 - s_t p_t) + \sum_{t=2}^{T} \sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k} p_t [p_{m(t,k)} - p_t] I_{p_{m(t,k)} \geqslant p_t} \tag{EC.A2}$$

In order to show that $f(\mathrm{P})$ is concave in $p \in PS_\Gamma$, it suffices to prove that for any fixed $\bar{\mathrm{P}} \in PS_\Gamma$, the tangent line at $f(\bar{\mathrm{P}})$ always lies above $f(\mathrm{P})$ for all P in PS_Γ. In Appendix B, we prove the following lemma:

LEMMA A.1. *Under any set of fixed ordering constraints* Γ, $f(\mathrm{P}) - f(\bar{\mathrm{P}}) \leqslant \nabla f(\bar{\mathrm{P}})(\mathrm{P} - \bar{\mathrm{P}})$ *for all* $\mathrm{P} \in PS_\Gamma$.

Therefore, $f(\mathrm{P})$ is concave over PS_Γ, so that for any particular set of fixed ordering constraints, the corresponding problem to determine an optimal price plan, P_Γ, becomes a concave maximization problem with the set of linear constraints as follows:

$$\text{Model (2)-}\Gamma \qquad Z_\Gamma^* = \max\left[\sum_{t=1}^{T} p_t(d_t^0 - s_t p_t) + \sum_{t=2}^{T} \sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k} p_t [p_{m(t,k)} - p_t]^+ - \sum_{t=1}^{T} c_t x_t - \sum_{t=1}^{T} h_t I_t\right]$$

$$\text{s.t.} \quad I_{t-1} = I_t + \sum_{t=1}^{T}(d_t^0 - s_t p_t) + \sum_{t=2}^{T} \sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[p_{m(t,k)} - p_t]^+ \quad \forall\, t = 1, \ldots, T$$

$$x_t \leqslant Q_t \quad \forall\, t = 1, \ldots, T$$
$$x_t, I_t, p_t \geqslant 0 \quad \forall\, t = 1, \ldots, T$$
$$(p_1, p_2, \ldots, p_T) \in PS_\Gamma$$

where the ordering constraints are represented by a set of linear inequalities by identifying which of constraints 2.3a, 2.3b, and 2.3c are binding. □

B. Proof of Lemma A.1

Proof. From Equation (EC.A1), $f(\mathrm{P}) - f(\bar{\mathrm{P}})$ can be expressed as follows:

$$f(\mathrm{P}) - f(\bar{\mathrm{P}}) = \sum_{t=1}^{T}(d_t^0[p_t - \bar{p}_t] - s_t[p_t^2 - \bar{p}_t^2]) + \sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[p_t p_{m(t,k)} - \bar{p}_t \bar{p}_{m(t,k)}] I_{p_{m(t,k)} \geqslant p_t}$$
$$- \sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[p_t^2 - \bar{p}_t^2] I_{p_{m(t,k)} \geqslant p_t}. \tag{EC.B1}$$

Noting that each residual revenue term, $\alpha_k^{t-k} s_{t-k} p_t p_{m(t,k)} I_{p_{m(t,k)} \geqslant p_t}$, in $f(\mathrm{P})$ will contribute to $\partial f/\partial p_t$ and $\partial f/\partial p_{m(t,k)}$, respectively, as follows:

$$\frac{\partial f}{\partial p_t} = \cdots + \alpha_k^{t-k} s_{t-k} p_{m(t,k)} I_{p_{m(t,k)} \geqslant p_t} + \cdots$$

and

$$\frac{\partial f}{\partial p_{m(t,k)}} = \cdots + \alpha_k^{t-k} s_{t-k} p_t I_{p_{m(t,k)} \geqslant p_t} + \cdots.$$

From this, we know that each term, $\alpha_k^{t-k} s_{t-k} p_t p_{m(t,k)} I_{p_{m(t,k)} \geqslant p_t}$, contributes two terms to $\nabla f(\bar{\mathrm{P}})(\mathrm{P} - \bar{\mathrm{P}})$ through

$$\alpha_k^{t-k} s_{t-k} \bar{p}_{m(t,k)} [p_t - \bar{p}_t] I_{p_{m(t,k)} \geqslant p_t} \qquad \text{and} \qquad \alpha_k^{t-k} s_{t-k} \bar{p}_t [p_{m(t,t)} - \bar{p}_{m(t,k)}] I_{p_{m(t,k)} \geqslant p_t}.$$

After some algebraic manipulation, we have

$$\nabla f(\bar{\mathrm{P}})(\mathrm{P} - \bar{\mathrm{P}}) = \sum_{t=1}^{T} \frac{\partial f(\bar{p}_t)}{\partial p_t}(p_t - \bar{p}_t)$$
$$= \sum_{t=1}^{T}(d_t^0[p_t - \bar{p}_t] - s_t[2\bar{p}_t p_t - 2\bar{p}_t^2]) - \sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[2\bar{p}_t p_t - 2\bar{p}_t^2] I_{p_{m(t,k)} \geqslant p_t}$$
$$+ \sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[\bar{p}_t p_{m(t,k)} + p_t \bar{p}_{m(t,k)} - 2\bar{p}_{m(t,k)} \bar{p}_t] I_{p_{m(t,k)} \geqslant p_t} \tag{EC.B2}$$

When we subtract (EC.B2) from (EC.B1) and cancel the appropriate terms, we obtain:

$$f(\mathrm{P}) - f(\bar{\mathrm{P}}) - \nabla f(\bar{\mathrm{P}})(\mathrm{P} - \bar{\mathrm{P}}) = -\sum_{t=1}^{T} s_t[p_t^2 - 2p_t\bar{p}_t + \bar{p}_t^2] - \sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[p_t^2 - 2p_t\bar{p}_t + \bar{p}_t^2] I_{p_{m(t,k)} \geqslant p_t}$$
$$+ \sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[p_t p_{m(t,k)} - \bar{p}_t p_{m(t,k)} + \bar{p}_{m(t,k)} \bar{p}_t - p_t \bar{p}_{m(t,k)}] I_{p_{m(t,k)} \geqslant p_t}$$

After completing the squares:

$$f(\mathrm{P}) - f(\bar{\mathrm{P}}) - \nabla f(\bar{\mathrm{P}})(\mathrm{P} - \bar{\mathrm{P}}) = -\sum_{t=1}^{T} s_t[p_t - \bar{p}_t]^2 - \sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[p_t - \bar{p}_t]^2 I_{p_{m(t,k)} \geqslant p_t}$$
$$+ \sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[(p_t - \bar{p}_t)(p_{m(t,k)} - \bar{p}_{m(t,k)})] I_{p_{m(t,k)} \geqslant p_t}.$$

We now combine the first and second terms, start the inner summation from $k=0$ by using the fact that $p_{m(t,0)} = p_t$ and the fact that $\alpha_0 = 1$, and split the combined summation term into two halves as follows:

$$\begin{aligned} f(\mathrm{P}) - f(\bar{\mathrm{P}}) - \nabla f(\bar{\mathrm{P}})(\mathrm{P} - \bar{\mathrm{P}}) &= -\frac{1}{2}\sum_{t=1}^{T}\sum_{k=0}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[p_t - \bar{p}_t]^2 I_{p_{m(t,k)} \geqslant p_t} \\ &\quad -\frac{1}{2}\sum_{t=1}^{T}\sum_{k=0}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[p_t - \bar{p}_t]^2 I_{p_{m(t,k)} \geqslant p_t} \\ &\quad +\sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[(p_t - \bar{p}_t)(p_{m(t,k)} - \bar{p}_{m(t,k)})] I_{p_{m(t,k)} \geqslant p_t} \qquad \text{(EC.B3)} \end{aligned}$$

We complete our proof by showing that each strictly positive term in the last summation is outweighed by two corresponding negative terms – one from the first summation and the other from the second summation.

For this, we first pair $-(1/2)\alpha_k^{t-k} s_{t-k}[p_t - \bar{p}_t]^2$ from the first summation term with the corresponding $\alpha_k^{t-k} s_{t-k}[(p_t - \bar{p}_t)(p_{m(t,k)} - \bar{p}_{m(t,k)})]$ from the last term. Since $m(t,k)$ is the period at which the minimum price is offered between period $t-k$ and $t-1$, if $t-k < m(t,k) \leqslant t-1$, there must be a period between period $t-k+1$ and period $t-1$ such that there exists the residual demand originating from period $t-k$ realized at price $\bar{p}_{m(t,k)}$, and thus the corresponding term, $-(1/2)\alpha_j s_{t-k}[p_{m(t,k)} - \bar{p}_{m(t,k)}]^2$ where $j = m(t,k) - (t-k)$ can be selected from the second summation. On the other hand, if $m(t,k) = t-k$, that is p_{t-k} is the minimum price for the next k period, a corresponding term, $-(1/2)\alpha_0 s_{t-k}[p_{m(t,k)} - \bar{p}_{m(t,k)}]^2$, must exist.

From the assumption that α_i are decreasing in i, we have

$$-\frac{1}{2}\alpha_j s_{t-k}[p_{m(t,k)} - \bar{p}_{m(t,k)}]^2 \leqslant -\frac{1}{2}\alpha_k^{t-k} s_{t-k}[p_{m(t,k)} - \bar{p}_{m(t,k)}]^2$$

for $0 \leqslant j \leqslant k$. Since every positive cross product term can be matched with two corresponding negative terms, a little algebra shows:

$$f(\mathrm{P}) - f(\bar{\mathrm{P}}) - \nabla f(\bar{\mathrm{P}})(\mathrm{P} - \bar{\mathrm{P}}) \leqslant -\frac{1}{2}\sum_{t=2}^{T}\sum_{k=1}^{\min(t-1,K)} \alpha_k^{t-k} s_{t-k}[p_t - \bar{p}_t - p_{m(t,k)} + \bar{p}_{m(t,k)}]^2 I_{p_{m(t,k)} \geqslant p_t} \leqslant 0, \quad \square$$

C. Proof of Lemma 2

Proof. Consider an n-period price sequence that the price in each of n period equals to the non-interaction optimal price

$$p_1 = p_2 = \cdots = p_n = p^*.$$

Then we can represent any feasible price charged over n periods as perturb p_i's around p^* by $(1+\delta_i(n))$. Then if we impose non-increasing structure on p_i, the profit function (4.1) can be written as follows:

$$\begin{aligned} f_n(\mathbf{p})|_{p_i=(1+\delta_i(n))p^*} &= \left[\sum_{i=1}^{n}(p_i - c)(d^0 - sp_i) + \alpha\sum_{i=2}^{n} s(p_{i-1} - p_i)(p_i - c)\right]_{p=p^*(1+\delta(n))} \\ &= -ncd^0 - \alpha csp^*(\delta_1(n) - \delta_n(n)) + \sum_{i=1}^{n} p^*(1+\delta_i(n))(d^o + cs - s(p^*(1+\delta_i(n)))) \\ &\quad + \alpha\sum_{i=2}^{n} s(p^*(1+\delta_{i-1}(n)) - p^*(1+\delta_i(n)))(p^*(1+\delta_i(n))) \end{aligned}$$

Substituting $d^0 + cs = 2sp^*$, we obtain:

$$\begin{aligned} f_n(\mathbf{p})|_{p_i=(1+\delta_i(n))p^*} &= -ncd^0 - \alpha csp^*(\delta_1(n) - \delta_n(n)) + \sum_{i=1}^{n} sp^{*2}(1+\delta_i(n))(2 - (1+\delta_i(n))) \\ &\quad + \alpha\sum_{i=2}^{n} sp^{*2}((1+\delta_{i-1}(n)) - (1+\delta_i(n)))(1+\delta_i(n)) \end{aligned}$$

$$
\begin{aligned}
&= -ncd^0 - \alpha csp^*(\delta_1(n) - \delta_n(n)) + \sum_{i=1}^{n} sp^{*2}(1-\delta_i^2(n))) \\
&\quad + \alpha \sum_{i=2}^{n} sp^{*2}(\delta_{i-1}(n) - \delta_i(n))(1+\delta_i(n)) \\
&= \sum_{i=1}^{n}(sp^{*2} - cd^0) - \alpha csp^*(\delta_1(n) - \delta_n(n)) - \sum_{i=1}^{n} sp^{*2}(\delta_i^2(n)) \\
&\quad + \alpha \sum_{i=2}^{n} sp^{*2}(\delta_{i-1}(n) - \delta_i(n))(1+\delta_i(n))
\end{aligned}
$$

Subtracting $f_n(\mathbf{p})|_{p_i=p^*}$ from $f_n(\mathbf{p})|_{p_i=(1+\delta_i(n))p^*}$ and dividing by the sequence length, n, we get the average additional profit per period when an n-period pricing sequence is used:

$$
\begin{aligned}
\Delta f_n &= \frac{sp^{*2}}{n}\left[\sum_{i=1}^{n} -(\delta_i(n))^2 + \alpha\sum_{i=2}^{n}(\delta_{i-1}(n) - \delta_i(n))(1+\delta_i(n)) - \alpha\frac{c}{p^*}(\delta_1(n) - \delta_n(n))\right] \\
&= \frac{sp^{*2}}{n}\Bigg([-(\delta_1(n))^2 - ((1+\alpha)\delta_2(n))^2 - \cdots - ((1+\alpha)\delta_n(n))^2] \\
&\qquad + \alpha[\delta_1(n)\delta_2(n) + \delta_2(n)\delta_3(n) + \cdots + \delta_{n-1}(n)\delta_n(n)]\alpha[\delta_1(n) - \delta_n(n)]\left(1-\frac{c}{p^*}\right)\Bigg) \qquad \text{(EC.C1)}
\end{aligned}
$$

Taking the derivative with respect to $\delta_i(n)$ and setting it to zero, we obtain:

$$\delta^*(n) = \mathbf{\Omega}_n^{-1}\cdot \mathbf{e}$$

where $\mathbf{\Omega}_n$ is an $n\times n$ matrix,

$$
\mathbf{\Omega}_n = \begin{pmatrix}
-2 & \alpha & 0 & \cdots & 0 & 0 \\
\alpha & -2(1+\alpha) & \alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \alpha & -2(1+\alpha)
\end{pmatrix}
$$

and $\mathbf{e}$ is an $n\times 1$ column vector,

$$
\mathbf{e} = \left[-\alpha\left(1-\frac{c}{p^*}\right) \quad 0 \quad \cdots \quad 0 \quad \alpha\left(1-\frac{c}{p^*}\right)\right]^T
$$

Solving for $\delta_i^*(n)$ leads to Equations (4.4)–(4.6), which is a sequence of non-increasing prices for n periods. From Lemma D.3, this price sequence must be decreasing and bounded above by d^0/s and below by c. The optimality of the sequence results from the fact that the solution satisfying Equations (4.4)–(4.6) is the unconstrained optimal solution of a concave function.

Noting that $p_i = p^*(1-\delta_i^*(n))$, $i = 1, 2, \ldots, n$ maximizes the average profit for n periods and, comparing this with the profit from no interaction pricing (i.e., $p_i = p^*$), the average profit under this policy increases by

$$
\begin{aligned}
\Delta f_n^* &= \frac{sp^{*2}\alpha}{n}\left[\frac{1}{2}(\mathbf{e}^T\mathbf{\Omega}_n^{-1}\mathbf{\Omega}_n\mathbf{\Omega}_n^{-1}\mathbf{e}) - (\mathbf{e}^T\mathbf{\Omega}_n^{-1}\mathbf{e})\right] \\
&= \frac{-sp^{*2}\alpha}{n}\left[\frac{1}{2}\mathbf{e}^T\mathbf{\Omega}_n^{-1}\mathbf{e}\right] \\
&= \frac{sp^{*2}\alpha}{n}\left[\frac{a}{2}(\delta_1^*(n) - \delta_n^*(n))\right]. \quad \square \qquad \text{(EC.C2)}
\end{aligned}
$$

D. Technical Lemmas for the Proof of Theorem 3

The following lemmas are used in the proof of Theorem 3, and are proved in the following subsections:

LEMMA D.1. *Let $r = 2 + 2/\alpha$ for all $n \geqslant 1$, $\beta_{i-1}(n)\beta_i(n) \geqslant r^i$, $i = 1, \ldots, n$.*

LEMMA D.2.
$$\lim_{n\to\infty}\frac{\beta_{n-1}(n)}{\beta_n(n)}=\frac{1}{r_1-r_\infty}\quad \text{where } r_\infty=\frac{1+\alpha-\sqrt{1+2\alpha}}{\alpha}\ \text{ and }\ r_1=\frac{2}{\alpha}.$$

LEMMA D.3.

$$\text{(i)}\quad \delta_1^*(n) \text{ is increasing in } n \text{ and } \lim_{n\to\infty}\delta_1^*(n)=\frac{a}{r_1-r_\infty} \tag{EC.D1}$$

$$\text{(ii)}\quad \delta_n^*(n)\geqslant\frac{-a}{r-1} \text{ for all } n\geqslant 1. \tag{EC.D2}$$

where $r_\infty=(1+\alpha-\sqrt{1+2\alpha})/\alpha$, $r_1=2/\alpha$, *and* $r=2+2/\alpha$.

D.1. Proof of Lemma D.1

PROOF. We prove this by induction. Clearly, it is true for $i=1$. Now, assume it is true for all $i<j$, and consider $i=j\leqslant n$.

$$\beta_{j-1}(n)\beta_j(n) = r\beta_{j-1}(n)^2-\beta_{j-1}(n)\beta_{j-2}(n) \tag{EC.D3}$$
$$= \beta_{j-1}(n)(r\beta_{j-1}(n)-\beta_{j-2}(n)) \tag{EC.D4}$$

Since $r=2+2/\alpha\geqslant 4$ and $\beta_{j-2}(n)\geqslant\beta_{j-3}(n)$,

$$\beta_{j-1}(n)\geqslant 4\beta_{j-2}(n)-\beta_{j-3}(n)\geqslant 3\beta_{j-2}(n)\quad \text{for } j\leqslant n. \tag{EC.D5}$$

Substituting and then applying induction hypothesis, we see that

$$\begin{aligned}\beta_{j-1}(n)\beta_j(n) &\geqslant \beta_{j-1}(n)(3r\beta_{j-2}(n)-\beta_{j-2}(n))\\ &\geqslant 3rr^{j-1}-r^{j-1}\\ &\geqslant r^j\end{aligned}$$

For $i=n$,

$$\begin{aligned}\beta_{n-1}(n)\beta_n(n) &= \beta_{n-1}(n)\left(\frac{2}{\alpha}\beta_{n-1}(n)-\beta_{n-2}(n)\right)\\ &\geqslant \beta_{n-1}(n)\left(\frac{6}{\alpha}\beta_{n-2}(n)-\beta_{n-2}(n)\right)\\ &\geqslant r^{n-1}\left(\frac{6}{\alpha}-1\right)\\ &\geqslant r^n\end{aligned}$$

The second last inequality comes from the fact that $(6/\alpha-1)\geqslant(2+2/\alpha)=r$ for $\alpha\in[0,1]$. □

D.2. Proof of Lemma D.2

PROOF. Consider a sequence, $\beta_i(n)$ for $1\leqslant i\leqslant(n-1)$. Using Lemma 2, we can calculate the ratio between $(i-1)$th and (i)th term in the sequence as follows:

$$q_1=\frac{\beta_0(n)}{\beta_1(n)}=\frac{1}{r},\ q_2=\frac{\beta_1(n)}{\beta_2(n)}=\frac{1}{\frac{\beta_2(n)}{\beta_1(n)}}=\frac{1}{\frac{r\beta_1(n)-\beta_0(n)}{\beta_1(n)}}=\frac{1}{r-\frac{1}{r}},\ldots,q_n=\frac{\beta_{n-1}(n)}{\beta_n(n)}$$

Simple algebraic manipulation shows that $\beta_i(n)$ is an increasing sequence and $\beta_i(n)$ increases by a factor of r for a sufficiently large n. Therefore, as $n\to\infty$, q_n converges to a limit and can be expressed as a continued fraction as follows:

$$\lim_{n\to\infty}q_n=\lim_{n\to\infty}\frac{\beta_{n-1}(n)}{\beta_n(n)}=\frac{1}{r-\frac{1}{r-\cdots}}. \tag{EC.D6}$$

Applying Lemma D.4 (below), the smaller root of polynomial $x^2 - rx + 1 = 0$ (defined by r_∞) is the limiting value of the continued fraction. Then, we have:

$$\lim_{n\to\infty} \frac{\beta_{n-2}(n)}{\beta_{n-1}(n)} = r_\infty = \frac{1+\alpha-\sqrt{1+2\alpha}}{\alpha} \tag{EC.D7}$$

Applying a little algebra, we get

$$\lim_{n\to\infty} \frac{\beta_{n-1}(n)}{\beta_n(n)} = \lim_{n\to\infty} \frac{\beta_{n-1}(n)}{2\beta_{n-1}(n) - \beta_{n-2}(n)} = \frac{\beta_{n-1}(n)}{r_1\beta_{n-1}(n) - r_\infty\beta_{n-1}(n)} = \frac{1}{r_1 - r_\infty} \quad \square \tag{EC.D8}$$

D.3. Lemma D.4

Lemma D.4. *Continued fraction* (EC.D6) *is convergent and its limiting value is the smaller root, r_∞ of polynomial $x^2 - rx + 1 = 0$, where $r = 2 + 2/\alpha$ and $0 < \alpha \leqslant 1$.*

Proof. Let u and v be two non-zero real numbers which satisfy the following conditions:

$$\begin{aligned} u \cdot v &= 1 \\ u + v &= r \end{aligned} \tag{EC.D9}$$

We can find such u and v by solving $x^2 - rx + 1 = 0$. If we substitute u and v into continued fraction (EC.D6) and divide it by u, we obtain the following expression:

$$\frac{Q}{u} = \frac{v}{u + v - \frac{uv}{u+v-\cdots}} \tag{EC.D10}$$

Dividing above and below the first fraction bar by u transforms this last expression into the equation:

$$\frac{Q}{u} = \frac{v/u}{1 + \frac{v}{u} - \frac{v}{u+v-\cdots}} \tag{EC.D11}$$

Equations (EC.D10) and (EC.D11) have the same sequence of convergents. Similarly, the terms above and below the second fraction bar can be divided by the u without changing the sequence of the convergents. In fact, this can be done above and below all of the fraction bars without changing the sequence of convergents. Hence, we obtain the following equivalent transformation of the (EC.D10):

$$\frac{Q}{u} = \frac{v/u}{1 + \frac{v}{u} - \frac{\frac{v}{u}}{1+\frac{v}{u}-\cdots}} \tag{EC.D12}$$

Let $x = u/v$ and consider the following expression:

$$\frac{1}{1 - \frac{Q}{u}} = \frac{1}{1 - \frac{x}{1+x-\frac{x}{1+x-\cdots}}} \tag{EC.D13}$$

The numerator and denominator of the convergents of a continued fraction satisfy a pair of recursions. If we write pth convergent of (EC.D13) by a_p/b_p, the recursions for a_p and b_p are:

$$\begin{aligned} a_{p+1} &= (1+x)a_p - xa_{p-1} \quad (p \geqslant 1) \\ b_{p+1} &= (1+x)b_p - xb_{p-1} \quad (p \geqslant 1) \end{aligned} \tag{EC.D14}$$

where $a_0 = 0$, $a_1 = 1$, $b_0 = 1$, and $b_1 = 1$. Equations (EC.D14) and mathematical induction imply that

$$\begin{aligned} a_p &= \sum_{j=0}^{p-1} x^j \quad (p \geqslant 2) \\ b_p &= 1 \qquad\quad (p \geqslant 2) \end{aligned} \tag{EC.D15}$$

If the pth convergent of continued fraction (EC.D10) is denoted by $q_p = \beta_{p-1}/\beta_p$, then the pth convergent of continued fraction (EC.D13) can be rewritten as follows:

$$\frac{a_p}{b_p} = \frac{1}{1 - q_{p-1}/u}. \tag{EC.D16}$$

Solving (EC.D16) for q_p and substituting (EC.D15) for a_p and b_p, we obtain:

$$q_p = u - \frac{u}{\sum_{j=0}^{p}\left(\frac{v}{u}\right)^j} \tag{EC.D17}$$

If we assume that u is smaller than v, then q_p converges to u in increasing fashion as $p \to \infty$ since $\sum_{j=0}^{p}(v/u)^j \to \infty$. Or if we assume that v is smaller than u, then $\sum_{j=0}^{p}(v/u)^j \to 1/(1-v/u)$ and q_p again converges to v in increasing fashion as $p \to \infty$ as shown below:

$$\begin{aligned}\lim_{p\to\infty} q_p &= u - \frac{u}{\frac{1}{1-\frac{v}{u}}} = u - \frac{u}{\frac{u}{u-v}} = u - (u - v)\\ &= v \quad \square\end{aligned} \tag{EC.D18}$$

D.4. Proof of Lemma D.3

Proof. From Lemma D.2

$$\delta_1^*(n) = \frac{\beta_{n-1}(n)\delta_0(n) - a}{\beta_n(n)}.$$

It can be easily shown that $\delta_1^*(n)$ increases as n increases by using the fact that both $\beta_{n-1}(n)/\beta_n(n)$ and $\beta_n(n)$ are monotone increasing. Furthermore, the convergence of a sequence, $\beta_{n-1}(n)/\beta_n(n)$ and the fact that $\lim_{n\to\infty}\beta_n(n) = \infty$ imply the existence of the limit. Replacing $\delta_0(n)$ with a as in Lemma D.2, we have:

$$\lim_{n\to\infty}\delta_1^*(n) = \lim_{n\to\infty}\frac{\beta_{n-1}(n)}{\beta_n(n)} = \frac{a}{r_1 - r_\infty}$$

To show the inequality, we consider the following system of difference equations:

$$\begin{aligned}\delta_1^*(n) &= \frac{\beta_{n-1}(n)}{\beta_n(n)}\delta_0(n) - \frac{1}{\beta_n(n)}a\\ \delta_2^*(n) &= \frac{\beta_{n-2}(n)}{\beta_{n-1}(n)}\frac{\beta_{n-1}(n)}{\beta_n(n)}\delta_0(n) - \frac{\beta_{n-2}(n)}{\beta_{n-1}(n)}\frac{1}{\beta_n(n)}a - \frac{1}{\beta_{n-1}(n)}a\\ &\ \vdots\\ \delta_n^*(n) &= \frac{\beta_0(n)}{\beta_n(n)}\delta_0(n) - a\left(\frac{\beta_0(n)}{\beta_{n-1}(n)\beta_n(n)} + \frac{\beta_0(n)}{\beta_{n-2}(n)\beta_{n-1}(n)} + \cdots + \frac{\beta_0(n)}{\beta_1(n)\beta_2(n)} + \frac{\beta_0(n)}{\beta_1(n)}\right)\end{aligned}$$

It is very easy to see that $\delta_i^*(n)$ is decreasing in i since the positive term in the above expression decreases while more negative term is added as i increases. To get a bound of $\delta_n^*(n)$, we take a close look at $\beta_i(n)\beta_{i-1}(n)$ terms. Lemma D.1 implies that $\beta_i(n)\beta_{i-1}(n)$ is bounded below by r^i and substituting the corresponding terms yields

$$\begin{aligned}\delta_n^*(n) &= \frac{\beta_0(n)}{\beta_n(n)}\delta_0(n) - a\left(\frac{\beta_0(n)}{\beta_{n-1}(n)\beta_n(n)} + \frac{\beta_0(n)}{\beta_{n-2}(n)\beta_{n-1}(n)} + \cdots + \frac{\beta_0(n)}{\beta_1(n)\beta_2(n)} + \frac{\beta_0(n)}{\beta_1(n)}\right)\\ &\geqslant \frac{\beta_0(n)}{\beta_n(n)}\delta_0(n) - a\left(\frac{\beta_0(n)}{r^n} + \frac{\beta_0(n)}{r^{n-1}} + \cdots + \frac{\beta_0(n)}{r^2} + \frac{\beta_0(n)}{r}\right)\\ &\geqslant -a\beta_0(n)\left(\frac{1}{r^n} + \frac{1}{r^{n-1}} + \cdots + \frac{1}{r^2} + \frac{1}{r}\right)\\ &\geqslant -a\beta_0(n)\left(\frac{1}{r-1}\right) = \frac{-a}{r-1}\end{aligned}$$

The second last inequality holds because $(\beta_0(n)/\beta_n(n))\delta_0(n)$ is nonnegative and the last equality comes from $\beta_0(n) = 1$. $\square$

E. Proof of Theorem 3

Proof. It follows immediately from Lemma 2 that the average profit increase under an n-period optimal pricing sequence depends only on $\delta_1^*(n)$ and $\delta_n^*(n)$, both of which are functions of a length of a pricing sequence n. We also note that $\delta_1^*(n)$ increases in n and converges to $a/(r_1 - r_\infty)$ while $\delta_n^*(n)$ is always bounded above by $-a/(r-1)$. Therefore,

$$\delta_1^*(n) - \delta_n^*(n) \leqslant a\left(\frac{1}{r_1 - r_\infty} + \frac{1}{r-1}\right) \quad \text{for all } n \geqslant 1.$$

Thus, the average profit increase for any n-period optimal pricing sequence is bounded above by some finite number $\Delta F_n^* = (a^2 sp^{*2}\alpha/(2n))[\alpha(3+\sqrt{(2\alpha+1)})/((1-\alpha+\sqrt{(2\alpha+1)})(\alpha+2))]$ as follows:

$$\begin{aligned}
\Delta f_n^* &= \frac{sp^{*2}\alpha a^2}{an}[\delta_1^*(n) - \delta_n^*(n)] \\
&\leqslant \frac{sp^{*2}\alpha}{n}\left[\frac{a^2}{2}\left(\frac{1}{r_1 - r_\infty} + \frac{1}{r-1}\right)\right] \\
&= \frac{a^2 sp^{*2}\alpha}{2n}\left[\frac{1}{r_1 - r_\infty} + \frac{1}{r-1}\right] \\
&= \frac{a^2 sp^{*2}\alpha}{2n}\left[\frac{\alpha\big(3+\sqrt{(2\alpha+1)}\big)}{\big(\big(1-\alpha+\sqrt{(2\alpha+1)}\big)(\alpha+2)\big)}\right] \\
&= \Delta F_n^*
\end{aligned} \tag{EC.E1}$$

On the other hand, the 2-period optimal price sequence and the 3-period optimal price sequence yield the following average profit increases, respectively:

$$\begin{aligned}
\Delta f_2^* &= \frac{asp^{*2}\alpha}{4}[\delta_1^*(2) - \delta_2^*(2)] \\
&= \frac{asp^{*2}\alpha}{4}\left[\frac{a\alpha(\alpha+2)}{4\alpha+4-\alpha^2} - \frac{a\alpha(\alpha-2)}{4\alpha+4-\alpha^2}\right] \\
&= \frac{a^2 sp^{*2}\alpha}{4}\left[\frac{4\alpha}{4\alpha+4-\alpha^2}\right]
\end{aligned} \tag{EC.E2}$$

$$\begin{aligned}
\Delta f_3^* &= \frac{a^2 sp^{*2}\alpha}{6}[\delta_1^*(3) - \delta_3^*(3)] \\
&= \frac{asp^{*2}\alpha}{6}\left[\frac{a\alpha(\alpha^2+4\alpha+2)}{2\alpha^2+8\alpha+4-\alpha^3} - \frac{a\alpha(\alpha^2-2\alpha-2)}{2\alpha^2+8\alpha+4-\alpha^3}\right] \\
&= \frac{a^2 sp^{*2}\alpha}{6}\left[\frac{6\alpha^2+4\alpha}{2\alpha^2+8\alpha+4-\alpha^3}\right]
\end{aligned} \tag{EC.E3}$$

Comparing Δf_2^* with Δf_3^* and the upper bound of the average profit increase under an n-period optimal pricing sequence, ΔF_n^* for $n \geqslant 4$, we obtain following inequalities:

$$\begin{aligned}
\Delta f_2^* = a^2 sp^{*2}\left[\frac{\alpha^2}{4\alpha+4-\alpha^2}\right] &\geqslant \Delta f_3^* = a^2 sp^{*2}\left[\frac{3\alpha^3+2\alpha^2}{6\alpha^2+24\alpha+12-3\alpha^3}\right] \\
&\geqslant \Delta F_4^* = a^2 sp^{*2}\left[\frac{\alpha^2\big(3+\sqrt{(2\alpha+1)}\big)}{8\big(\big(1-\alpha+\sqrt{(2\alpha+1)}\big)(\alpha+2)\big)}\right] \\
&\geqslant \Delta F_n^* = a^2 sp^{*2}\left[\frac{\alpha^2\big(3+\sqrt{(2\alpha+1)}\big)}{2n\big(\big(1-\alpha+\sqrt{(2\alpha+1)}\big)(\alpha+2)\big)}\right] \quad \text{for } n \geqslant 5
\end{aligned} \tag{EC.E4}$$

Therefore, the two-period optimal pricing sequence maximizes the average profit. □

F. Proof of Last Part of Theorem 2

PROOF. By Property 1, every pricing plan with k regeneration points can be decomposed into $k+1$ non-increasing pricing sequences. We proved in Theorem 3 that the optimal revenue increase is generated by repeating the 2-period optimal pricing sequence, which implies that if the planning period T is an even number then the pricing plan is simply repeating the 2-period optimal pricing sequence $T/2$ times. If T is an odd number, then it is enough to show that $2\Delta f_2^* \leqslant 3\Delta f_3^*$, since by inequality (EC.E4) these are the two most profitable ways to create an odd length pricing plan.

$$\begin{aligned}\frac{3\Delta f_3^*}{2\Delta f_2^*} &= \frac{\frac{3}{6}a^2 sp^{*2}\alpha\left[\frac{6\alpha^2+4\alpha}{2\alpha^2+8\alpha+4-\alpha^3}\right]}{\frac{2}{4}a^2 sp^{*2}\alpha\left[\frac{4\alpha}{4\alpha+4-\alpha^2}\right]}\\ &= \frac{1}{2} * \frac{(3\alpha+2)(4\alpha+4-\alpha^2)}{(2\alpha^2+8\alpha+4-\alpha^3)}\\ &\geqslant 1 \quad \text{for all } 0<\alpha\leqslant 1 \quad \square\end{aligned} \tag{EC.F1}$$

.

G. Proof of Lemma 3

We divide the proof into two parts. In the first part, we obtain an unconstrained optimal solution which maximizes Equation (5.2). In the second part, we then show that the solution from the first part is in fact a sequence of decreasing prices, and satisfies all constraints.

First Part of Proof

The proof is very similar to the proof for the 1-period interaction case. Let $\mathbf{p}$ be an n-period pricing sequence. Substituting p_i with $p_i = (1+\delta_i(n))p^*$, the profit function (5.2) can be rewritten as follows:

$$\begin{aligned} f_n^K(\mathbf{p})|_{p_i=(1+\delta_i(n))p^*} &= \left[\sum_{i=1}^n (p_i-c)(d^0-sp_i) + \alpha s\sum_{i=2}^n \min(i-1,K)(p_{i-1}-p_i)(p_i-c)\right]_{p=p^*(1+\delta(n))}\\ &= -ncd^0 - \alpha csp^*\left(\sum_{i=1}^{\min(n-1,K)} \delta_i(n) - \min(n-1,K)\delta_n(n)\right)\\ &\quad + \sum_{i=1}^n p^*(1+\delta_i(n))(d^o+cs-s(p^*(1+\delta_i(n))))\\ &\quad + \alpha s\sum_{i=2}^n \min(i-1,K)(p^*(1+\delta_{i-1}(n))-p^*(1+\delta_i(n)))(p^*(1+\delta_i(n)))\end{aligned}$$

Substituting $d^0+cs=2sp^*$, we obtain:

$$\begin{aligned} f_n^K(\mathbf{p})|_{p_i=(1+\delta_i(n))p^*} &= -ncd^0 - \alpha csp^*\left(\sum_{i=1}^{\min(n-1,K)} \delta_i(n) - \min(n-1,K)\delta_n(n)\right)\\ &\quad + \sum_{i=1}^n sp^{*2}(1+\delta_i(n))(2-(1+\delta_i(n))) + \alpha s\sum_{i=2}^n p^{*2}\min(i-1,K)(\delta_{i-1}(n)-\delta_i(n))(1+\delta_i(n))\\ &= -ncd^0 - \alpha csp^*\left(\sum_{i=1}^{\min(n-1,K)} \delta_i(n) - \min(n-1,K)\delta_n(n)\right) + \sum_{i=1}^n sp^{*2}(1-\delta_i(n)^2))\\ &\quad + \alpha s\sum_{i=2}^n p^{*2}\min(i-1,K)(\delta_{i-1}(n)-\delta_i(n))(1+\delta_i(n))\\ &= \sum_{i=1}^n (sp^{*2}-cd^0) - \alpha csp^*\left(\sum_{i=1}^{\min(n-1,K)} \delta_i(n) - \min(n-1,K)\delta_n(n)\right) - \sum_{i=1}^n sp^{*2}(\delta_i(n)^2)\\ &\quad + \alpha s\sum_{i=2}^n p^{*2}\min(i-1,K)(\delta_{i-1}(n)-\delta_i(n))(1+\delta_i(n)).\end{aligned} \tag{EC.G1}$$

We then subtract $f_n^K(\mathbf{p})|_{p_i=p^*}$ from $f_n^K(\mathbf{p})|_{p_i=(1+\delta_i(n))p^*}$ and divide this by n to get the average profit increase per period:

$$\begin{aligned}
\Delta f_n^K &= \frac{sp^{*2}}{n}\bigg(\sum_{i=1}^{n} -(\delta_i(n))^2 + \alpha\sum_{i=2}^{n}\min(i-1,K)(\delta_{i-1}(n)-\delta_i(n))(1+\delta_i(n)) \\
&\qquad - \alpha\frac{c}{p^*}(\delta_1(n)+\cdots+\delta_{\min(n-1,K)}(n)-\min(n-1,K)\delta_n(n))\bigg) \\
&= \frac{sp^{*2}}{n}\bigg([-(\delta_1(n))^2-(1+\alpha)\delta_2(n)^2-(1+2\alpha)\delta_3(n)^2-\cdots-(1+\min(n-1,K)\alpha)\delta_{\min(n-1,K)}(n)^2] \\
&\qquad + \alpha[\delta_1(n)\delta_2(n)+2\delta_2(n)\delta_3(n)+\cdots+\min(n-1,K)\delta_{n-1}(n)\delta_n(n)] \\
&\qquad + \alpha[\delta_1(n)+\cdots+\delta_{\min(n-1,K)}(n)-\min(n-1,K)\delta_n(n)]\left(1-\frac{c}{p^*}\right)\bigg)
\end{aligned}$$

Taking the derivative with respect to $\delta_i(n)$ and setting it to zero, we obtain:

$$\mathbf{\Omega}_n^K \cdot \delta^*(n) = \mathbf{e}^K \tag{EC.G2}$$

where $\mathbf{\Omega}_n^K$ is an $n\times n$ matrix:

$$\mathbf{\Omega}_n^K = \begin{pmatrix}
-2 & \alpha & 0 & 0 & \cdots & 0 & 0 \\
\alpha & -2(1+\alpha) & 2\alpha & 0 & \cdots & 0 & 0 \\
0 & 2\alpha & -2(2+\alpha) & 3\alpha & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \min(n-1,K)\alpha & -2(1+\min(n-1,K)\alpha)
\end{pmatrix},$$

$\mathbf{e}^K$ is an $n\times 1$ column vector:

$$\mathbf{e}^K = \begin{cases}
[-\alpha a \quad -\alpha a \quad \cdots \quad -\alpha a \quad \alpha(n-1)a]^T, & \text{if } n\leqslant K+1; \\
[-\alpha a \quad -\alpha a \quad \cdots \quad -\alpha a \quad 0 \quad \cdots \quad 0 \quad \alpha K a]^T, & \text{if } n > K+1,
\end{cases}$$

and $a=(1-c/p^*)$. We note that only the first K terms and the last term in vector $\mathbf{e}^K$ are non-zero for $n>K+1$.

Dividing both sides of (EC.G2) by α, we obtain:

$$\mathbf{\Omega}_n^K = \begin{pmatrix}
-2\alpha^{-1} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & -2-2\alpha^{-1} & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 & -4-2\alpha^{-1} & 3 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \min(n-2,K) & -2\min(n-2,K)-2\alpha^{-1} & \min(n-1,K) \\
0 & 0 & 0 & 0 & \cdots & 0 & \min(n-1,K) & -2\min(n-1,K)-2\alpha^{-1}
\end{pmatrix}$$

and

$$\mathbf{e}^K = \begin{cases}
[-a \quad -a \quad \cdots \quad -a \quad (n-1)a]^T, & \text{if } n\leqslant K+1; \\
[-a \quad -a \quad \cdots \quad -a \quad 0 \quad \cdots \quad 0 \quad Ka]^T, & \text{if } n > K+1
\end{cases}$$

Solving (EC.G2) for $\delta_i^*(n)$ leads to Equations (5.4) for $n\leqslant K+1$ and (5.6) for $n>K+1$, respectively.

Second Part of Proof

We now show that an n-period price sequence in Lemma 3 is feasible: that is, p_i's are non-increasing, and bounded above by d^0/s and below by c. Let $p_i=(1+\delta_i^*(n))p^*$ for $i=1,\ldots,n$. Then, it suffices to show

$$\frac{d^0}{s} \geqslant (\delta_1^*(n)+1)p^* \geqslant (\delta_2^*(n)+1)p^* \geqslant (\delta_3^*(n)+1)p^* \geqslant \cdots \geqslant (\delta_n^*(n)+1)p^* \geqslant c.$$

Plugging $p^* = (d^0 + sc)/2s$ and $a = 1 - c/p^*$, the above inequality can be rewritten as

$$a = 1 - \frac{c}{p^*} \geqslant \delta_1^*(n) \geqslant \delta_2^*(n) \geqslant \delta_3^*(n) \geqslant \cdots \geqslant \delta_n^*(n) \geqslant -a = -\left(1 - \frac{c}{p^*}\right).$$

We prove this in three steps. First, we show that $\delta_i^*(n)$'s are monotonically non-decreasing assuming $\delta_1^*(n) \leqslant a/(2\alpha^{-1} - 1) \leqslant a$. We then show that $\delta_1^*(n)$ must be bounded above by $a/(2\alpha^{-1} - 1)$, therefore by a as well. We complete the proof by showing that $\delta_n^*(n) \geqslant -a$ independent of the sequence length n.

Step 1: Proof of $\delta_1^*(n) \geqslant \delta_2^*(n) \geqslant \cdots \geqslant \delta_n^*(n)$. In this step, we show that $\delta_i^*(n)$'s are non-increasing if $\delta_1^*(n) \leqslant a/(2\alpha^{-1} - 1)$. We divide the proof into two cases: $n \leqslant K + 1$ and $n > K + 1$

Case 1. $n \leqslant K + 1$

We first recall the equations of (EC.G2):

$$-2\alpha^{-1}\delta_1^*(n) + \delta_2^*(n) = -a, \quad \text{for } i = 1 \tag{EC.G3}$$

$$(i-1)\delta_{i-1}^*(n) - 2(i - 1 + \alpha^{-1})\delta_i^*(n) + (i)\delta_{i+1}^*(n) = -a, \quad \text{for } i = 2, \ldots, n-1 \tag{EC.G4}$$

$$(n-1)\delta_{n-1}^*(n) - 2(n - 1 + \alpha^{-1})\delta_n^*(n) = (n-1)a, \quad \text{for } i = n. \tag{EC.G5}$$

We obtain $\delta_i^*(n)$, $i = 1, \ldots, n$ by solving a system of Equations (EC.G3), (EC.G4), and (EC.G5), and without loss of generality, we write $\delta_i^*(n)$ in the following form:

$$\delta_i^*(n) = a\left(\frac{1}{2\alpha^{-1} - 1} - r_i\right) = a\left(\frac{\alpha}{2 - \alpha} - r_i\right) \quad \text{for some } r_i \tag{EC.G6}$$

Note that $r_1 \geqslant 0$ from our assumption that $a/(2\alpha^{-1} - 1)$.

Consider (EC.G3):

$$-2\alpha^{-1}\delta_1^*(n) + \delta_2^*(n) = -a$$

Substituting the corresponding expressions of (EC.G6) for $\delta_1^*(n)$ and $\delta_2^*(n)$, we obtain:

$$-2\alpha^{-1}a\left(\frac{1}{2\alpha^{-1} - 1} - r_1\right) + a\left(\frac{1}{2\alpha^{-1} - 1} - r_2\right) = -a$$

After some algebra, we get $2\alpha^{-1}(r_1) = r_2$; thus, $r_2 \leqslant r_1$ and $\delta_1^*(n) \geqslant \delta_2^*(n)$. Now suppose $r_1 \leqslant r_2 \cdots \leqslant r_i$ for some i (i.e., $\delta_k^*(n)$ is decreasing for $k \leqslant i$). We substitute $\delta_{i-1}^*(n)$, $\delta_i^*(n)$, and $\delta_{i+1}^*(n)$ in $(i-1)\delta_{i-1}^*(n) - 2(i - 1 + \alpha^{-1})\delta_i^*(n) + i\delta_{i+1}^*(n) = -a$ (EC.G4) with the corresponding expressions from (EC.G6) to get

$$-\frac{i-1}{i}r_{i-1} + \frac{i - 1 + \alpha^{-1}}{i}r_i + \frac{i - 1 + \alpha^{-1}}{i}r_i = r_{i+1} \tag{EC.G7}$$

The induction hypothesis and the fact that $\alpha \leqslant 1$ imply that $0 \leqslant r_i \leqslant r_{i+1}$, thus $\delta_i^*(n) \leqslant \delta_{i+1}^*(n)$ for all $i = 1 \ldots n - 1$.

Case 2. $n > K + 1$

As in the previous case, suppose $a/(2\alpha^{-1} - 1) \geqslant \delta_1^*(n)$. Recall that $\delta_i^*(n)$, $i = 1, \ldots, n$ are a solution to the system of equations (EC.G2) as follows:

$$-2\alpha^{-1}\delta_1^*(n) + \delta_2^*(n) = -a, \quad \text{for } i = 1, \tag{EC.G8}$$

$$(i-1)\delta_{i-1}^*(n) - 2(i - 1 + \alpha^{-1})\delta_i^*(n) + (i)\delta_{i+1}^*(n) = -a, \quad \text{for } i = 2, \ldots, K, \tag{EC.G9}$$

$$K\delta_{i-1}^*(n) - 2(K + \alpha^{-1})\delta_i^*(n) + K\delta_{i+1}^*(n) = 0, \quad \text{for } i = K + 1, \ldots, n - 1, \tag{EC.G10}$$

$$K\delta_{n-1}^*(n) - 2(K + \alpha^{-1})\delta_n^*(n) = Ka, \quad \text{for } i = n \tag{EC.G11}$$

Equations (EC.G8) and (EC.G9) are identical to their counterparts in the previous case and, from this, it is easy to see that $\delta_1^*(n) \geqslant \delta_2^*(n) \geqslant \cdots \geqslant \delta_K^*(n)$. Therefore, it suffices to show that

$$\delta_K^*(n) \geqslant \delta_{K+1}^*(n) \geqslant \delta_{K+2}^*(n) \geqslant \cdots \geqslant \delta_n^*(n).$$

Let

$$\delta_i^*(n) = \delta_{i-1}^*(n) - v_i \quad \text{for } i = K + 1 \cdots n. \tag{EC.G12}$$

Using the fact that $\delta_1^*(n) \geqslant \delta_2^*(n) \ldots \delta_{K-1}^*(n) \geqslant \delta_K^*(n)$ in Equation (EC.G9) for $i=K$, $\delta_K^*(n) \geqslant \delta_{K+1}^*(n)$ and $v_{K+1} = \delta_K^*(n) - \delta_{K+1}^*(n) \geqslant 0$. Now suppose $v_i \geqslant v_{i-1} \geqslant \cdots \geqslant v_{K+1} \geqslant 0$ for some i, $K+1 \leqslant i \leqslant n-1$. Substituting the corresponding expressions into Equations (EC.G10) and (EC.G11), we get:

$$K(\delta_i^*(n)+v_i) - 2(K+\alpha^{-1})\delta_i + K(\delta_i^*(n) - v_{i+1}) = 0 \quad \text{for } i = K+1 \cdots n.$$

Applying some algebra, this can be further simplified to

$$-\frac{2\alpha^{-1}}{K}\delta_i^*(n) + v_i = v_{i+1} \quad \text{for } i = K+1\cdots n. \tag{EC.G13}$$

If $\delta_i^*(n) \leqslant 0$, $v_{i+1} \geqslant 0$ (thus $\delta_i^*(n) \geqslant \delta_{i+1}^*(n)$ from Equation (EC.G13)). On the other hand, if $\delta_i^*(n) \geqslant 0$, (5.6) implies that

$$\delta_{i+1}^*(n) = \frac{K\beta_{n-i-1}(n)}{\beta_{n-i}(n)}\delta_i(n) - \frac{a}{\beta_{n-i}(n)}K^{n-i}. \tag{EC.G14}$$

Then, from Lemma G.1, presented and proved below in §G.1, (i.e., $0 \leqslant \beta_{n-i}(n)/\beta_{n-i+1}(n) \leqslant 1/K$), we have

$$\begin{aligned}\delta_i^*(n) &= \frac{K\beta_{n-i}(n)}{\beta_{n-i+1}(n)}\delta_{i-1}^*(n) - \frac{a}{\beta_{n-i+1}(n)}K^{n-i+1}\\ &\leqslant \frac{K\beta_{n-i}(n)}{\beta_{n-i+1}(n)}\delta_{i-1}^*(n)\\ &\leqslant \delta_{i-1}^*(n).\end{aligned} \tag{EC.G15}$$

This implies $v_{i+1} \geqslant 0$. From induction, $\delta_i^*(n)$ is non-increasing in i.

Step 2: Proof of $\delta_1^*(n) \leqslant a$.

Case 1. ($n \leqslant K+1$) Suppose $\delta_1^*(n) > a/(2\alpha^{-1}-1)$. Then, $r_1 < 0$ from (EC.G6). Furthermore, from (EC.G7),

$$0 > r_1 > r_2 > \cdots > r_n \qquad \text{and} \qquad 0 < \delta_1^*(n) < \delta_2^*(n) < \cdots < \delta_n^*(n). \tag{EC.G16}$$

On the other hand, consider Equation (EC.G5):

$$(n-1)\delta_{n-1}^*(n) - 2(n-1+\alpha^{-1})\delta_n^*(n) = (n-1)a$$

and solve for $\delta_n^*(n)$:

$$\delta_n^*(n) = \frac{n-1}{2(n-1+\alpha^{-1})}\delta_{n-1}^*(n) - a\frac{n-1}{2(n-1+\alpha^{-1})} \tag{EC.G17}$$

If $\delta_{n-1}^*(n) > 0$, the above equation implies that $\delta_n^*(n) < \delta_{n-1}^*(n)$, and this contradicts (EC.G16). Therefore, $r_1 > 0$ and $\delta_1^*(n) \leqslant a/(2\alpha^{-1}-1)$.

Case 2. ($n > K+1$) Again, suppose $\delta_1^*(n) > a/(2\alpha^{-1}-1)$. From recursive relations (EC.G8)–(EC.G11), we have

$$\delta_n^*(n) > \delta_{n-1}^*(n) > \cdots > \delta_1^*(n) > \frac{a}{2\alpha^{-1}-1} > 0 \tag{EC.G18}$$

On the other hand, consider Equation (EC.G11)

$$K\delta_{n-1}^*(n) - 2(K+\alpha^{-1})\delta_n^*(n) = Ka$$

Rearranging terms, we get

$$\delta_n^*(n) = \frac{K}{2(K+\alpha^{-1})}\delta_{n-1}^*(n) - a\frac{K}{2(K+\alpha^{-1})}$$

If $\delta_{n-1}^*(n) > 0$, $\delta_n^*(n) < \delta_{n-1}^*(n)$. This contradicts (EC.G18), hence $\delta_1^*(n) > a/(2\alpha^{-1}-1)$.

Step 3: Proof of $\delta_n^*(n) \leqslant -a$. We now prove that $\delta_n^*(n) \geqslant -a$. Let $\delta_i^*(n) = a(1/(2\alpha^{-1}-1) - r_i)$, $i = 1, \ldots, n$. Then it suffices to show $r_n \leqslant 2/(2-\alpha)$. Rewriting (EC.G5) (for $n \leqslant K+1$) and (EC.G11) (for $n > K+1$), we have

$$\min(n-1,K)\delta_{n-1}^*(n) - 2(\min(n-1,K)+\alpha^{-1})\delta_n^*(n) = \min(n-1,K)a \tag{EC.G19}$$

Now substituting $\delta_i^*(n) = a(1/(2\alpha^{-1}-1) - r_i)$ for $i = n-1$ and n into Equation (EC.G19), we obtain:

$$r_n + \frac{\min(n-1,K)}{(\min(n-1,K)+2\alpha^{-1})}(r_n - r_{n-1}) = \frac{\min(n-1,K)2\alpha^{-1} + 2\alpha^{-1}}{(2\alpha^{-1}-1)(\min(n-1,K)+2\alpha^{-1})} \tag{EC.G20}$$

Since $\delta_i^*(n)$ is nonincreasing (i.e., $r_n - r_{n-1} \geqslant 0$)

$$r_n \leqslant \frac{(\min(n-1,K)2\alpha^{-1}+2\alpha^{-1})}{(2\alpha^{-1}-1)(\min(n-1,K)+2\alpha^{-1})} \leqslant \frac{2\alpha^{-1}}{2\alpha^{-1}-1} - \frac{2\alpha^{-1}}{\min(n-1,K)+2\alpha^{-1}} \leqslant \frac{2}{2-\alpha}.$$

G.1. Proof of Lemma G.1

LEMMA G.1. *Assume that* $n > K + 1$:

$$\frac{\beta_i}{\beta_{i-1}} \geqslant K$$

where $i = 1 \cdots n - K$.

PROOF. Note that $\beta_1(n)/\beta_0(n) \geqslant K$ since $\beta_0(n) = 1$ and $\beta_1(n) = 2(K + \alpha^{-1})$. Suppose $\beta_i(n)/\beta_{i-1}(n) \geqslant K$ for some $1 \leqslant i < n - K$. From Equation (5.6), we have

$$\beta_{i+1}(n) = 2(K + \alpha^{-1})\beta_i(n) - (K)^2\beta_{i-1}(n)$$

Applying induction hypothesis, we get

$$\begin{aligned}\frac{\beta_{i+1}(n)}{\beta_i(n)} &= 2(K + \alpha^{-1}) - (K)^2\frac{\beta_{i-1}(n)}{\beta_i(n)}\\ &= 2(K + \alpha^{-1}) - \frac{K^2}{\beta_{i-1}(n)/\beta_{i-2}(n)} \geqslant 2(K + \alpha^{-1}) - \frac{K^2}{K} \geqslant K.\end{aligned}$$

A similar argument is used for $i \geqslant n - K$. Thus, $\beta_{i+1}(n)/\beta_i(n) \geqslant K$ for all $i < n$. $\square$